PREFACE

In a bid to standardize higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific, generic elective, ability and skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry their acquired credits. I am happy to note that the university has been recently accredited by National Assessment and Accreditation Council of India (NAAC) with grade "A".

UGC (Open and Distance Learning Programmes and Online Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the six semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University
Under Graduate Degree Programme
Choice Based Credit System (CBCS)

Subject : Honours In Physics (HPH)

Course : Mechanics

Course Code : GE-PH-11

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Unit 1 ❐ Vectors

Structure

1.1 Objectives

This unit helps you to develop on idea about vector and its properties. Also you will learn vector algebra & application in daily life.

1.2 Introductions

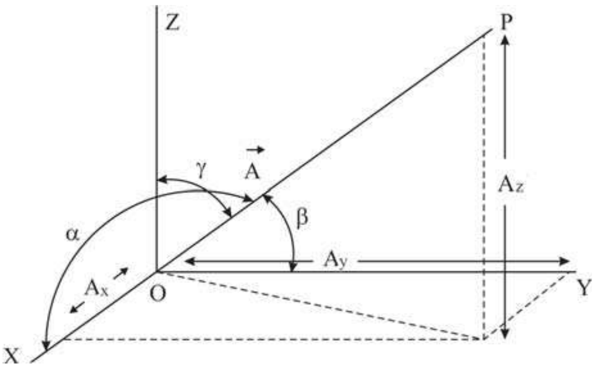
The Physical quantities we often come acrors are of two types. Those which have magnitudes only such as mass, volume etc are known as scalar, whereas those which have magnitude and direction are vectors. Force, velocity, acceleration etc are some very common vectors.

We write a vector $\vec{A}$ having a magnitude $|\vec{A}|$ or simply A and geometrically represented by a line segment of length proportional to A and whose direction is along $\vec{A}$. In onthogonal rectangular cartesian system the vector $\vec{A}$ is expressed as

$$\vec{A} = \hat{i}\ A_x + \hat{j}\ A_y + \hat{k}\ A_z$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the axes and A_x, A_y and A_z are components of $\vec{A}$ along the axes. The magnitude of the vector in terms of its components is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\ ,\ \text{where}$$

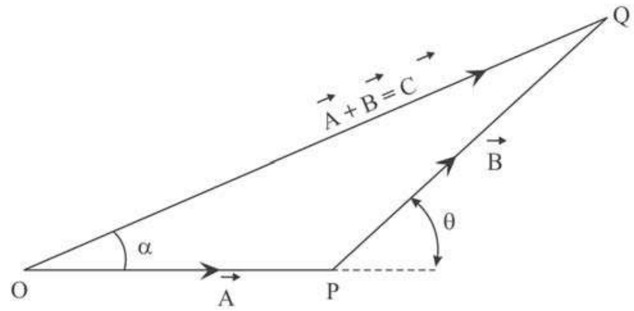


A_x = A cos α, A_y = A cos β and A_z = A cos γ; where α, β and γ are the angles between the vector and the three co-ordinate axes. cos α, cos β and cos γ are the direction cosines of the vector and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

1.3 Vector Algebra

Sum of two vectors : Sum of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A}+\vec{B}=\vec{C}$ and is geometrically represented as following.



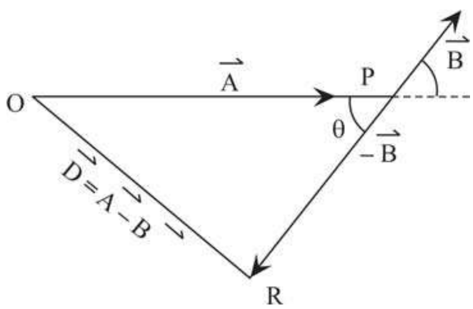
We draw a line OP along the direction of $\vec{A}$ and with magnitude OP proportional to A. $\vec{O}P$ represents $\vec{A}$. From the end point P of $\vec{A}$ we draw a line PQ along the direction $\vec{B}$ and of magnitude proportional to B. $\vec{P}\theta$ represents the vector $\vec{B}$. $\vec{O}Q$, the line joining. The start point of $\vec{A}$ to the end point of $\vec{B}$ is the vector $\vec{C}$ represents the Sum of $\vec{A}$ and $\vec{B}$ in direction and magnitude.

From simple geometry the magnitude of the vector sum is $C = \sqrt{A^2 + B^2 + 2AB\cos\theta}$ where θ is angle made by $\vec{B}$ with $\vec{A}$. The vector $\vec{C}$ will make angle α with $\vec{A}$ given by

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

In case of difference

$$\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$$



We draw $O\vec{P}=\vec{A}$. From P we draw PR proportional to B but opposite to the direction of $\vec{B}$. $P\vec{R}$ represents – $\vec{B}$. The line joining O to R is the vector $\vec{A}-\vec{B}$ in direction and magnitude.

$$D^2 = A^2 + B^2 - 2AB\ cos\theta$$

Scalar or dot product

The Scalar or a dot product of two vectors is scalar quantity equal to the product of magnitudes of the two vectors and the cosine of angle between them.

Thus

$$\vec{A}.\vec{B} = AB\cos\theta = \left(\hat{i}\ A_x + \hat{j}\ Ay + \hat{k}\ A_z\right)\cdot\left(\hat{i}\ A_x + \hat{j}\ Ay + \hat{k}\ A_z\right)$$

$$= A_xB_x + A_yB_y + A_zB_z;\ \ \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1;\ \ \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{j}$$

The work down W by a force $\vec{F}$ to make a displacement $\vec{S}$ is given by

$$W = \vec{F}.\vec{S} = FS\cos\theta$$

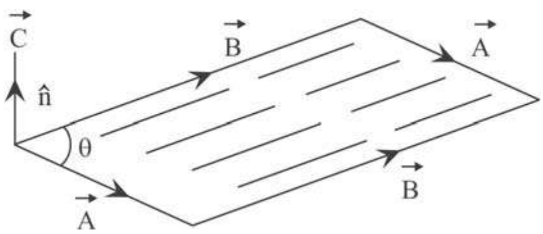
is a scalar quantity equal to the product of force (F) and the displacement in the direction of the force $S\cos\theta$.

1.4 Vector product or Cross product

A vector product or a cross product of two linearly independent vectors expressed as $\vec{A}\times\vec{B} = \vec{C}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$ and therefore normal to the plane containing $\vec{A}$ & $\vec{B}$, such that when a right hand screw is rotated from $\vec{A}$ to $\vec{B}$ the direction of motion of the screw head gives the direction of the product and the magnitude is equal to the product of the magnitudes of two vectors and the sine of the angle between them. We write

$$\vec{C} = \vec{A}\times\vec{B} = A\,B\sin\theta\,\hat{n}$$

where $\hat{n}$ is the unit vector along $\vec{C}$ and θ is the angle between the vectors $\vec{A}$ and $\vec{B}$.



The cross product $\vec{C}$ of $\vec{A}$ and $\vec{B}$ gives the area of the parallelogram with vectors $\vec{A}$ and $\vec{B}$ as sides.

In terms of components–

$$\vec{A}\times\vec{B} = \left(\hat{i}\ A_x + \hat{j}\ A_y + \hat{k}\ A_z\right)\times\left(\hat{i}\ B_x + \hat{j}\ B_y + \hat{k}\ B_z\right)$$

$$= \hat{i}\left(A_yB_z - B_yA_z\right) + \hat{j}\left(A_zB_x - B_zA_x\right) + \hat{k}\left(A_xB_y - B_xA_y\right)$$

$$\therefore\ i \times i = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \text{and} \quad \begin{matrix} \hat{i} & \times & \hat{j} & = \hat{k} \\ \hat{j} & \times & \hat{k} & = \hat{i} \\ \hat{k} & \times & \hat{i} & = \hat{j} \end{matrix}$$

Obviously $\bar{B} \times \hat{A} = -\hat{A} \times \hat{B}$

The torque $\vec{N}$ i.e. the moment of force $\vec{F}$ and $\vec{r}$ is the position vector of point of application of force then,

$$\vec{N} = \vec{r} \times \vec{F}$$

Derivative of a vector with respect to a parameter :

We consider a vector

$$\vec{A} = \hat{i}\ A_x + \hat{j}\ A_y + \hat{k}\ A_z$$

The derivative of the vector $\vec{A}$ with respect to a parameter α say is defined as.

$$\frac{d\vec{A}}{d\alpha} = \underset{\Delta\alpha \to 0}{Lt} \frac{\vec{A}(\alpha + \Delta\alpha) - \vec{A}(\alpha)}{\Delta\alpha}$$

$$= \underset{\Delta\alpha \to 0}{Lt} \left[\frac{A_x(\alpha + \Delta\alpha) - A_x(\alpha)}{\Delta\alpha} \hat{i} + \frac{A_y(\alpha + \Delta\alpha) - A_y(\alpha)}{\Delta\alpha} \hat{j} + \frac{A_z(\alpha + \Delta\alpha) - A_z(\alpha)}{\Delta\alpha} \hat{k} \right]$$

$$= \frac{d\ A_x}{d\alpha} \hat{i} + \frac{dA_y}{d\alpha} \hat{j} + \frac{dA_z}{d\alpha} \hat{k}$$

The time derivative of velocity $\vec{v}$ w.r.t *t* is

$$\frac{d\vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k}$$

which is the acceleration

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}\ .$$

$$\frac{dv_x}{dt} = a_x\ ;\ \frac{dv_x}{dt} = a_x\ ;\ \frac{dv_z}{dt} = a_z$$

v_x, v_y and v_z are components of velocity and a_x, a_y and a_z are components of acceleration along the co-ordinate areas.

Solved problems

(1) Find the angle between the vectors $\vec{A} = 4\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{B} = \left(9\hat{i} - 6\hat{j} + 3\hat{k}\right)$.

Solution : From the definition of dot products, if θ be the angle between the vectors,

$$\vec{A}.\vec{B} = A\,B\cos\theta = \left(A_x B_x + A_y B_y + A_z B_z\right)$$

or $$\cos\theta = \frac{4\times 9 + 2\times(-6) + 6\times 3}{\sqrt{4^2 + 2^2 + 6^2}\,\sqrt{9^2 + (-6)^2 + 3^2}}$$

$$= \frac{42}{\sqrt{56}\,\sqrt{126}} = \frac{42}{\sqrt{2\times 4\times 7}\,\sqrt{7\times 2\times 3\times 3}}$$

$$= \frac{42}{7\times 2\times 2\times 3} = \frac{42}{84} = \frac{1}{2}$$

$$= \cos\,60^o$$

$$\therefore \quad \theta = 60^o$$

2. Find the area of a parallelogram whose sides are determined by the vectors $\vec{A} = \hat{i} - \hat{j} + 3\hat{k}$ and $2\hat{i} - 7\hat{j} + \hat{k}$

The area of the parallelogram $\Delta = \left|\vec{A}\times\vec{B}\right|$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & h \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$= \hat{i}(-1+21) + \hat{j}(6-1) + \hat{k}(-7+2)$$

$$= 20\,\hat{i} + 5\hat{j} - 5\hat{k}$$

$$\Delta = \left|\vec{A}\times\vec{B}\right| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{450} = 15\sqrt{2}$$

(3) Find the unit vector along $\vec{P} = 5\hat{i} - 3\hat{j} + 7\hat{k}$

The unit vector $= \dfrac{\vec{P}}{|\vec{P}|} = \dfrac{5\hat{i} - 3\hat{j} + 7\hat{k}}{\sqrt{5^2 + (-3)^2 + 7^2}}$

$$= \frac{5\hat{i} - 3\hat{j} + 7\hat{k}}{\sqrt{25 + 9 + 49}} = \frac{1}{\sqrt{83}}\left(5\hat{i} - 3\hat{j} + 7\hat{k}\right)$$

(4) Find the dot product of $\vec{a} = 4\hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} + 5\hat{j} + 4\hat{k}$

The dot product is

$$\vec{a}.\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

$$= 4 \times 2 + 3 \times 5 + 7 \times 4$$

$$= 8 + 15 + 28 = 51$$

1.5 Exercises

(1) Find the vector product of $\vec{P} = 4\hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{Q} = 2\hat{i} + 5\hat{j} + 4\hat{k}$.

(2) Find the component of $\vec{A} = 3\hat{i} + \hat{j} + 4\hat{k}$ in the direction of $\vec{B} = \hat{i} - \hat{j} + \hat{k}$.

(3) Show the vector $\vec{A} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{B} = -9\hat{i} - 7\hat{j} - 15\hat{k}$ are perpendicular to each other.

(4) Vector $\vec{A} = 2\hat{i} + 6\hat{j} + 27\hat{k}$ and vector $\vec{B} = \hat{i} + \lambda\hat{j} + \mu\hat{k}$ are parallel. Find the values of λ and μ.

(5) If $\vec{P} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{Q} = \hat{i} + 3\hat{j} - 5\hat{k}$

Show that $\vec{P} + \vec{Q}$ and $\vec{P} - \vec{Q}$ are mutually perpendicular.

(6) Using vector prove that area of a triangle is equal to $\frac{1}{2} ab \sin\theta$, where a and b are two adjacent sides and θ is the angle between them.

Unit 2 ❒ Ordinary Differential Equation

Structure

2.1 Objectives

By reedy this unit you will learn about different kind of differential equation & its technique of solving such equations.

2.2 Introductions

An equation involving unknown function of an independent variable and the derivatives of the function is known as differential equation. It is a very useful device to solve problems in physics. In applications, the function or the functions usually represents physical quantities, the derivatives represent their rate of change and the differential equation defines a relation between them. If y(t) represents a function of an independent variable 't' and $\frac{dy}{dt}$ is its derivative, then

$$F\left(y, \frac{dy}{dt}, t\right) = 0$$

is a ordinary differential equation. The differential equation is ordinary if it does not involve partial derivatives.

The order of a differential equation is determined by the term with the highest derivative.

$$y + \frac{dy}{dx} = 5x$$

is a first order linear differential equation. First order since only $\frac{dy}{dx}$ not $\frac{d^2y}{dx^2}$ or $\frac{d^3y}{dx^3}$ appears in the equation.

Linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives.

$$\frac{d^2y}{dx^2}+kx=0 \qquad (2) \qquad k = \text{constant}$$

is a second order linear differential equation.

A differential equation is homogeneous if once all the terms involving unknown functions are collected together on one side of the equation, the other side is identically zero, Equation (2) is a second order linear homogeneous equation.

A first order differential equation is said to be homogeeous if it may be written as :

$$f(x, y)\ dy = g\ (x, y)\ dx$$

where f and g are homogeneous function of the same degree of x and y. We introduce an variable $v=\frac{y}{x}$.

$$f(x\ dv + v\ dx) = g\ dx$$

or $$(g - vf\ )\ dx = x\ dv\ f$$

$$\frac{dx}{x}=\frac{f}{g-vf}\,dv=h\,(v)\,dv$$

which can be solved by integration.

Solving a first order linear differential equation using integrating factor.

We take a linear differential equation in the form

$$\frac{dy}{dx}+P(x)y=Q(n) \qquad (1)$$

multiply both sides of equation (1) by a + ve function $\mu\,(x)$, that transforms the LHS into the derivative of the product μy

$$\mu\frac{dy}{dx}+\mu\,Py=\mu Q \qquad (2)$$

As per the choice of μ

$$\text{LHS} = \mu\frac{dy}{dx}+y\frac{d\mu}{dx} = \frac{d}{dx}(\mu y)$$

equation (2) becomes

$$\frac{d}{dx}(\mu y) = \mu Q$$

$$\mu y = \int \mu\, Q\, dx$$

or $$y = \frac{1}{\mu}\int \mu\, Q\, dx \qquad (3)$$

Eqa (3) is the solution of eqn (1) in terms of $\mu(x)$ and $Q(x)$. $\mu(x)$ is called the integrating factor of eqn. (1).

From the condition imposed on μ.

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + P\mu\, y$$

or, $$\mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + P\mu\, y$$

or, $$\frac{d\mu}{dx} = P_{\mu}, \quad \mu = e^{\int p dx}$$

μ is called integrating factor of eqn (1) because its presence makes the equation integrable,

Solve the equation :

$$x\frac{dy}{dx} = x^2 + 3, \ (1) \quad x>0,$$

We put the eqn. in standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x \quad ; \qquad P(x) = -\frac{3}{x},\ Q(x) = x$$

The integrating factor is

$$\mu(x) = e^{\int P dx} = e^{\int -\frac{3}{x}dx}$$

$$= e^{-3\ln(|x|)}$$

$$= e^{-3\ln x} \quad x > 0$$

$$= \frac{1}{x^3}$$

The solution is

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\, Q(x)\, dx$$

$$= x^3 \int \frac{x}{x^3}\, dx = x^3 \int \frac{1}{x^2}\, dx$$

$$= x^3 \left(-\frac{1}{x} + c \right); \quad c = \text{constant}$$

or $\quad y = -x^2 + cx^3$.

2.3 Separable Equation

A differential equation is said to be separable if the variables can be separated and can be written as

$$F(y)\ dy = G(x)\ dx$$

once this is done, all that is needed to solve that equation is to integrate both sides,

Let us solve the equation

$$x\ dx + \text{see}\ x \sin y\ dy = 0 \qquad (1)$$

We rewrite eqn (1) as

$$\text{See}\ x \sin y\ dy = -x\ dx$$

or $\quad \sin y\ dy = -x \cos dx$

Integrating both sides.

$$-\cos y = -\left[x \sin x - \int \sin x\ dx \right]$$

$$= -(x \sin x + \cos x + c)$$

$\therefore$ the required Soln. is

$$\cos y = x \sin x + \cos x + c\ ; \quad c = \text{constant}$$

Exact differential equation

A first order differential equation (of one variable) is called exact or exact differential if it is the result of a simple differentiation.

The equation

$$P(x, y)\ dx + Q\ (x, y)\ dy = 0$$

is exact if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

In this case there will be function $R(x, y)$ such that

$$\frac{\partial R}{\partial x} = P \text{ and } \frac{\partial R}{\partial y} = Q.$$

$R(x, y)$ is the solution of the problem.

Solve the differential equation by exact differential.

$$e^y\ dx + (2y + xe^y)\ dy = 0. \qquad (1)$$

Comparing with the standard equation

$$P(x, y)\ dx + Q(x, y)\ dy = 0 \qquad (2)$$

we get

$$P(x, y) = e^y \quad \text{and} \quad Q(x, y) = 2y + xe^y$$

$$\frac{\partial Q}{\partial x} = e^y;\ \frac{\partial P}{\partial y} = e^y.$$

i.e. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, hence the given equation is exact.

To find the soln $R(x, y)$ we write

$$\frac{\partial R}{\partial x} = e^y \qquad (3)$$

and $$\frac{\partial R}{\partial y} = 2y + x\,e^y \qquad (4)$$

from 3.

$$R(x, y) = \int e^y \, dx = x\, e^y + \varphi(y) \tag{5}$$

from (5)

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\left[xe^y + \varphi(y) \right]$$

$$= x\, e^y + \varphi'(y) \tag{6}$$

equating the rhs of (6) and (4) we have.

$$x\, e^y + \varphi'(y) = 2y + xe^y$$

or $\qquad \varphi'(y) = 2y$

$$\varphi'(y) = y^2$$

From (5)

$$R(x, y) = xe^y + y^2$$

The general solution is

$$xe^y + y^2 = c.$$

Second order linear homogeneous equation with constant coefficients are of great value in physical problems. As an illustration we discuss damped harmonic vibration.

Suppose a mass m is subjected to a restoring force proportional to displacement ($-\mu x$) and a resistive force proportional to instantaneous velocity ($-k\dot{x}$), the equation of motion can be written as.

$$m\ddot{x} = -\mu x - k\dot{x}$$

or $\qquad \ddot{x} + \frac{k}{m}\dot{x} + \frac{\mu}{m}x = 0$

or $\qquad \ddot{x} + 2b\dot{x} + w^2 x = 0 \qquad (1) \qquad$ where $\frac{k}{m} = 2b$

and $\frac{\mu}{m} = w^2$

Let $x = e^{pt}$ be a trial solution,

$$\dot{x} = px \quad \text{and} \quad \ddot{x} = p^2 x$$

putting this in eqn (1) we get.

$$p^2 + 2bp + w^2 = 0$$

or $$p = -b \pm \sqrt{b^2 - w^2}$$

has two values say $\alpha = -b + \sqrt{b^2 - w^2}$

and $\beta = -b - \sqrt{b^2 - w^2}$

and the solution of the problem can be witten as.

$$x = Ae^{\alpha x} + Be^{\beta x}, \qquad \text{A and B are constants.}$$

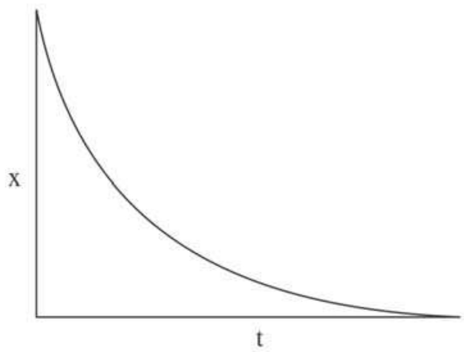
We discuss three different causes.

Case 1

$b^2 > w^2$ i.e large resistive force.

$$x = Ae^{bt} e^{\sqrt{\left(b^2 - w^2\right)t}} + Be^{-bt} e^{-\sqrt{b^2 - w^2}} t$$

both the terms gradually decreases to zero with time. There is no oscillation. The motion of the mass is shown in the figure. This motion is known as over damped motion.



Case 2.

$b^2 < w^2$, Small resistive face.

We write the soln as

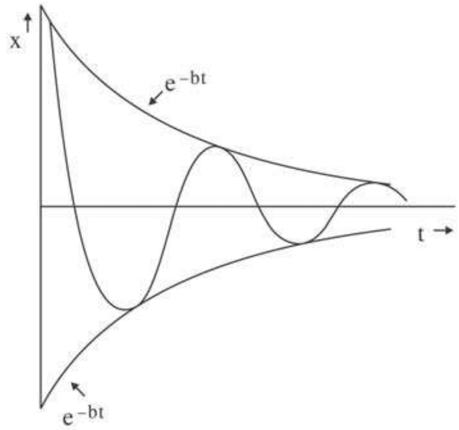
$$x = e^{-bt}\left(Ae^{i\gamma t} + Be^{-i\gamma t}\right); \quad \gamma = \sqrt{\omega^2\ b^2}$$

$$= e^{-bt}\left(c_1 \cos \gamma t + c_2 \sin \gamma t\right)$$

$$= e^{-bt}\ D \cos(\gamma t - \delta) \qquad [c_1,\ c_2 \text{ and } \delta \text{ are constants}]$$

The motion is an oscilatory one. Its amplitude De^{-bt} is exponentially decreasing with time. This is known as damped vibration.

Case 3.



$$b^2 = w^2, \ \alpha = \beta$$

In this case the solution of the differential equation is

$$x = (c + Dt)\, e^{-bt}$$

This motion is known as critical damped motion. It is critical in the sense that if $b>w$ the motion is non-oscillatory and if $b < w$ the motion is oscillatory.

(1) Solve the differential equation $\frac{dy}{dx} = 6y^2x$ given that $y(1) = \frac{1}{25}$

$$\frac{dy}{dx} = 6y^2x$$

or $\quad y^{-2}\ dy = 6x\ dx$

Integrating

$$\int y^{-2}dy = \int 6x\, dx$$

or $\quad -\frac{1}{y} = 3x^2 + c, \qquad c = \text{constant.}$

putting $y = -\frac{1}{25}$ and $x = 1$, we get

$$-\frac{1}{1/25} = 3.1^2 + c$$

or $\quad C + 3 = -25$

$\quad C = -28$

∴ The required solution is

$$-\frac{1}{y} = 3x^2 - 28$$

or $\quad y = \dfrac{1}{28 - 3x^2}$

(2) Show that $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx}$ has a solution

$y = c_1 + c_2e^{2x}$, where c_1 and c_2 are constant.

we have

$$y = c_1 + c_2e^{2x}$$

$$\frac{dy}{dx} = 2c_2e^{2x}$$

$$\frac{d^2y}{dx^2} = 4c_2e^{2x} = 2\frac{dy}{dx}$$

Thus, $y = c_1 + c_2e^{2x}$ is a solution of

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx}.$$

(3) Find the solution of $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{2y - 4}$

given that $y(1) = 3$.

We rewrite the differential equation as

$$dy(2y - 4) = (3x^2 + 4x - 4)\,dx.$$

or $\quad 2ydy - 4dy = 3x^2\,dx + 4x\,dz - 4dx$

Integrating both sides separately we have

$$y^2 - 4y = x^3 + \frac{4}{2}x^2 - 4x + c,$$

or $\quad$ with $y = 3$ at $x = 1$, we get.

$$3^2 - 4 \times 3 = 1^2 + 2.1^2 - 4.1 + c$$

or $9 - 12 = 1 + 2 - 4 + c$

or, $-3 = -1 + c$,

or, $c = -2$

$\therefore$ $y^2 - 4y = x^3 + 2x^2 - 4x - 2$ is the required solution.

2.4 Exercises

(1) State the order and degree of the following differential equations.

i) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - 3x + 2y = 8$; (2, 1)

ii) $\left(\frac{dy}{dx}\right)^5 - 2x = 3\sin x - \sin y$; (1,5)

iii) $\left(\frac{d^2y}{dx^2}\right)^4 + 2\left(\frac{dy}{dx}\right)^7 - 57 = 3$ (2,4)

(2) Find the general solution of the differential equation dy + 7x dx = 0. Obtain the particular solution with $y(o) = 2$.

(3) Establish the differential equation of a simple harmonic motion and obtain its general solution.

Unit 3 ❒ Laws of Motion

Structure

3.1 Objectives

By this unit you will gather knowledge about motion of objects & its relevant lows. An Idiea of centre of mass & its motion will also develop. Finally you will learn about Rocket motion in detail.

3.2 Introductions

Newton's Laws of motion describe the relationship between a body and the forces acting upon it and its motion in response to those forces. Three laws known as Newton's laws of motion together laid the foundation of classical mechanics.

First Law :– The state of motion of a body does not change until and external force is applied on it.

Second Law :– The rate of change of linear momentum of a body is proportional to the external force applied on it.

Third Law :– Every action has an equal and opposite reaction.

First law was actually present in topics of scientific discussion as Galelio's Law of Inertia before the statement by Newton was made. The law introduced the concept of force as an agent responsible for the change of state of motion of a body. While the 1st Law gives a qualitative definition of force, the second Law gives a quantitative statement.

The linear momentum $\vec{p}$ of a body is the product of its mass 'm' and velocity $\vec{v}$

$$\vec{p} = m\vec{v}$$

The second law mathematically may be written as

$$\frac{d}{dt}(m\vec{v}) \propto \vec{F}\,,$$ $\vec{F}$ is total external force applied on the body.

If we confine to the cases where m = constant, then

$$\vec{F} = k\,m\frac{d\vec{v}}{dt}$$ k = a constant of proportionality.

or $$\vec{F} = k\,m\vec{a}$$ $\vec{a} = \frac{d\vec{v}}{dt}$ = rate of change of volocity or acceleration.

If the force that produce unit acceleration on unit mass is taken as unit of force k becomes unity and we have

$$\vec{F} = m\vec{a} \tag{1}$$

The force that produce 1m s^{-2} of acceleration on 1 kg of mass is 1 Newton (IN).

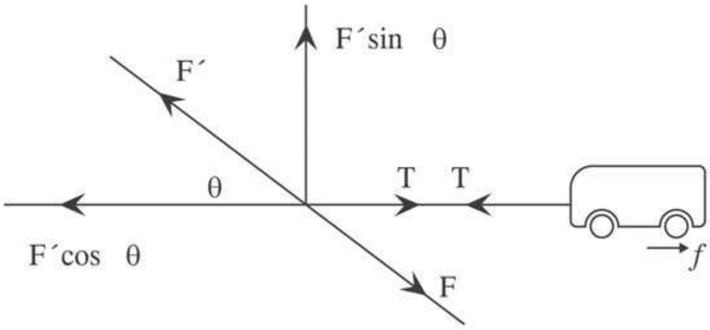
From eqn. (1)

$$m = \frac{F}{a} \quad \text{or} \quad a = \frac{F}{m}$$

i.e. greater is the value of m more it is difficult to produce acceleration i.e. more it is difficult to make the change of state of motion. Mass is therefore defined as inertia of the body.

Again if F = 0, a = 0 i.e. no force means no acceleration. No acceleration means no change of state of motion. This statement is nothing but the statement of first law of motion.

According to the third law of motion no force in nature is alone but always accompanied with reaction. This law plays a very important role in our daily lie. As an illustration we explain how a horse draw a cart.



The horse with his hooves applies an oblique force F on the ground. The ground in reply, applies a force of reaction F´ (F´ = F). F´ has an upward vertical component F´ sin θ which balances the weight of the horse. Tension T is created in the rope. The frictional force f of the ground is applied on the cart in backward direction. If $T > f$ the cart will move forward. If $F' \cos\theta > T$ the horse will move forward.

The equation of motion of the horse is

$$F' \cos\theta - T = Ma \qquad (1)\ ;\ M = \text{Mass of the horse}$$

The equation of motion of the cart is

$$T - f = ma, \qquad (2)\ ;\ a = \text{acceleration of the system}$$

$$m = \text{mass of the cart}$$

Adding equation (1) and equation (2) we get the equation of motion of the horse and cart composite system as

$$F' \cos\theta - f = (M + m)\ a$$

The equal and opposite force of action and reaction can never cancel each other since they act on different bodies.

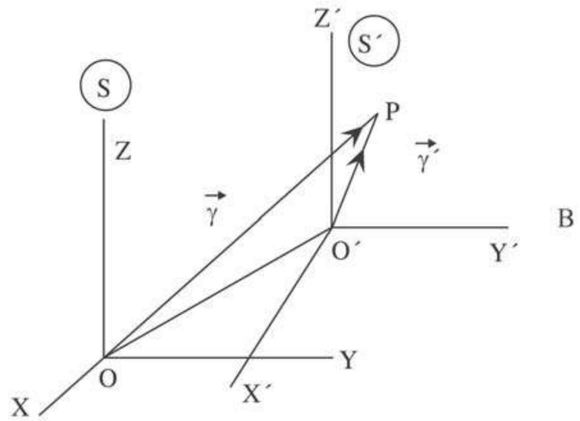
3.3 Frame of Reference

A framework that is used for observation and description of physical phenomenon and formulation of Physical laws usually consists of an observer, a coordinate system and a clock to assign time at a position with respect to the coordinate system is a frame of reference. Proper choice of frame of reference makes the study of motion of the body easy and solution of the problem less complicated.

To an observer in a running train all co-passangers are at rest but the objects outside train are in motion. On the other hand to an observer standing in the station, all passenger in the train are moving while the trees and land are at rest.

3.3.1 Inertial frame of Reference

We consider two orthogonal Cartesian coordinate systems (S) and (S′) as two frames of references. The coordinate axes are parallel. The origin O of s is at rest but o′ of s′ is moving with uniform velocity v with respect to 0. $\vec{r}$ and $\vec{r}'$ are position of a point particle P at a time t and $OO' = \vec{v}t$. (At $t = 0$, O and O′ are taken to be coincident) obviously



$$\vec{r} = \vec{r}' + \vec{v}t$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{v}$$

and $$\ddot{\vec{r}} = \ddot{\vec{r}}' + 0 \qquad (1)$$

Newton's second law of motion for the point particle P of mass m (say) in s frame is

$$m\ddot{\vec{r}} = \vec{F} \qquad (2)$$

From equation (1) and (2) we write

$$m\ddot{\vec{r}}' = \vec{F} \qquad (3)$$

Equation (2) and (3) are of exactly same form. The reference from s at rest, is known as inertial frame and Newton's Laws are valid in this frame. Equation (3) states that Newton's Laws are also valid in s' frame. In fact all frames moving with uniform velocity with respect to an inertial frame are also inertial. Two other facts are very relevent from this discussion that uniform velocity and rest are not distinguishable and the state of motion of an inertial frame can not be determined by an observer in the frame.

3.3.2 Non-inertial frame of Reference

Let the S' frame is moving with uniform acceleration $\vec{a}$ and at $t = 0$. O and O' are coincident and $v = 0$. At time t.

$$\vec{r} = \vec{r}' + OO'$$

$$= \vec{r}' + \frac{1}{2}\vec{a}t^2$$

$$\dot{\vec{r}} = \dot{\vec{r}}' + \vec{a}t$$

and $$\ddot{\vec{r}} = \ddot{\vec{r}}' + \vec{a} \qquad (4)$$

The Newton's second law of motion looks in *s* frame like

$$m\ddot{\vec{r}} = \vec{F} \qquad (5)$$

with the help of eqn. (4) and (5) the equation of motion of the particle *m* S' frame becomes.

$$m\ddot{\vec{r}}' + m\vec{a} = \vec{F}$$

or $$m\ddot{\vec{r}}' = \vec{F} - m\vec{a} \qquad (6)$$

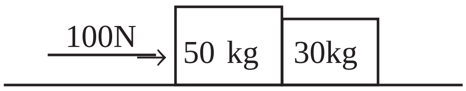
Though the left hand side of equation (6) contains the product of mass and acceleration right hand side contains over and above the applied force, another force term equal to negative of the product of mass of the particle and acceleration of the frame of reference. Equation (6) is different from Newton's law. Thus Newton's laws are not valid in such accelerating frames. Such frames are known as non-inertial frames. Along with the externally applied free $\vec{F}$ another force term $-m\vec{a}$ appears in the equation of motion. This force is due only for the acceleration of the frame. This force is not originated due to interaction of some physical quantities and only originated due to acceleration of frame. Such force of reference frame vanishes when viewed from an intertial frame, is known as pseudo force or ficticious force'. We realise this force in our daily life when we move in a lift. When the lift is moving uniformly the man in it experiences only his own weight. When the lift is going upward with an acceleration 'a' his apparant weight become mg + ma he feels heavier and when the lift is going down-ward with an acceleration 'a' his apparant weight become ms – ma and he feels lighter. In case the suspension cord breaks down and he suflers free fall, a = g, the man feels weightless.

A rotation is an acceleration. A body at rest in a rotating frame of reference feels a pseudo force known as centrefugal force.

When a car moving with uniform motion changes its direction of motion a man sitting in it feels a force opposite to the direction of turning. A boy riding a marry go-round feels a force drawing him radially outward. These are contrefugal forces whose value is $mr\,\omega^2$ where *m* is the mass of the body, *r* is the distance of the body from the centre of rotation and ω is the magnitude angular velocity of the rotating frame.

If a body in a rotating frame is not at rest but has a velocity, over and above the centrefugal force it feels another pseudo force known as coreolis force perpendicular to the velocily of the body and the axis of rotation of the frame.

(1) Two blocks of mass 50 kg and 30 kg are in contact on a smooth surface. A force of 100 N is applied on the 50 kg block as shown.



Find the acceleration of the system and the contact force between the blocks.

Solution : If f is the acceleration, then

$$100 = (50 + 30) \times f.$$

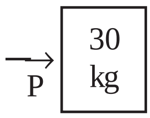
$$\text{or} \qquad f = \frac{100}{80} = 12.5 \text{ ms}^{-2}$$

If P be the contact force, the force diagram for the 50 kg mass is

$$100 - P = 50 \times 1.25 = 62.5$$

$$\therefore \quad P = 100 - 62.5 = 37.5 \text{ N}$$

The force diagram for 30 kg mass is



$$P = 30 \times 1.25$$
$$= 37.5 \text{ N.}$$

(2) A 75 kg man stands on a platform scale in an elevator. Standing from rest the elevator ascendes, attains its maximum speed of 1.20 m/s´ in ls. It travels with this constant speed for next 10s. The elevator then undergoes a retardation for 1.7s and comes to rest. What does the scale register?

i) before the elevator starts to move ?

ii) during the 1st 1 sec ?

iii) While elevator is travelling at constant speed?

iv) during the time it is slowing down ? ($g = 10 \text{ ms}^{-2}$)

Solution :

mg

R

i) There is no acceleration $\therefore$ R = mg

$= 75 \times 10$

$= 750$ N

ii) during the 1st s. the acceleration $f = \dfrac{1.2}{1}$

$= 1.2 \text{ ms}^{-2}$

$R - mg = mf$

$R = mg + mf = 75 \times 10 + 75 \times 1.2$

$= 750 + 90 = 840$ N.

iii) Since acceleration is zero

R – ms = 0, R = mg = 750 N

iv) If the retardation is a

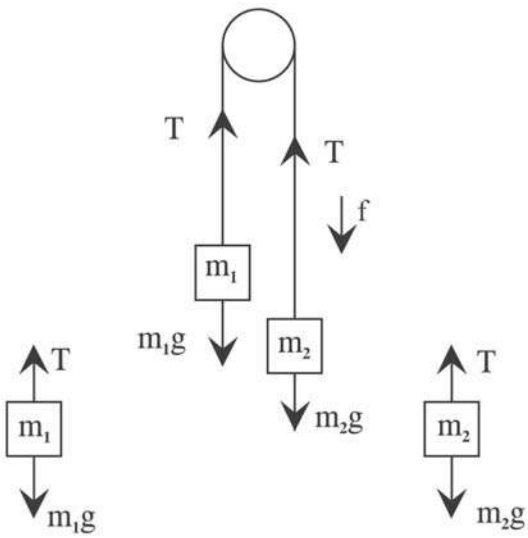
$1.2 = a \times 1.7$; $a = \dfrac{1.2}{1.7} = 0.7$

mg – R = ma

R = mg – ma = 750 – 75 × .7

= 750 – 52.5 = 697.5 N.

(3) A system of two masses m_1 and m_2 capable of moving through a frictionless pully *P* connected with a string is known as Atwood Machine. Compute the acceleration of the masses and tension of the string.



Let us suppose that m_2 is moving down with acceleration *f* and m_1 is going up with same acceleration and *T* is the tension of the string. For the mass m_1 we write

$$T - m_1g = m_1f \quad \ldots\ldots. (1)$$

For the mass m_2 we write

$$m_2 g - T = m_2 f \qquad \text{.......} (2)$$

Adding equation (1) and (2)

$$m_2 g - m_1 g = m_1 f + m_2 f$$

or $$f = \frac{(m_2 - m_1)g}{m_1 + m_2}$$

From (1) $$T = m_1 g + m_1 f = m_1 g\left[1 + \frac{m_2 - m_1}{m_2 + m_1}\right]$$

$$= \frac{2m_1 m_2 g}{m_1 + m_2}$$

(4) A 20 kg object has a velocity $4\hat{i}\ ms^{-1}$ at $t = 0$. A constant force $\left(2\hat{i} + 4\hat{j}\right)$ N Acts on the object for 3*s*. What is the magnitude of velocity at the end of 3*s* ?

Solution :

For *x*– motion $u_x = 4ms^{-1}$, $F_x = 2N$.

acceleration $f_x = \frac{2}{20} = 0.1\ \mathrm{ms}^{-2}$

$$v_x = u_x t + \frac{1}{2} f_x t^2 = 4 \times 3 + \frac{1}{2} \times 0.1 \times 9$$

$$= 12 + 0.45 = 12.45\ \mathrm{ms}^{-1}$$

For Y - motion, $u_y = 0$, $F_y = 4$ N.

acceleration $f_y = \frac{4}{20} = 0.2\ \mathrm{ms}^{-2}$

$$v_y = 0 + \frac{1}{2} \times 0.2 \times 9 = 0.9\ \mathrm{ms}^{-1}$$

$\therefore$ The magnitude of velocity at the end of 3*s*. is

$$v = \sqrt{(12.45)^2 + (0.9)^2}$$

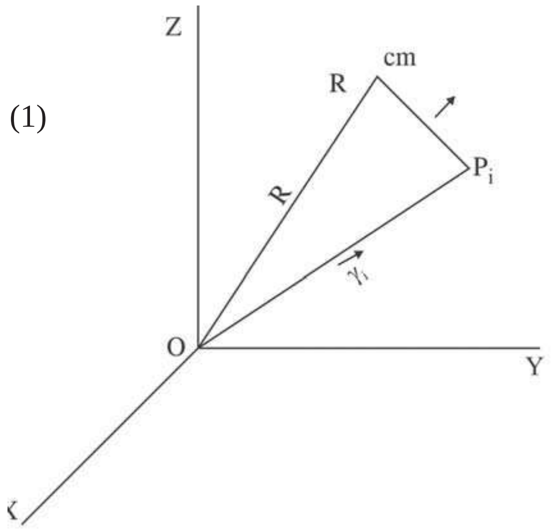
$$= \sqrt{155.0025 + .81} \quad = \sqrt{155.8125}$$

$$= 12.48 \text{ ms}^{-1}$$

3.4 Dynamics of System of Particles Centre of Mass

The centre of mass of a distribution of mass in space is the unique point where the weighted relative position of the distributed mass sums to zero. This is the point to which a force may be applied to cause linear acceleration without any angular acceleration. Calculations in mechanics are often simplified when formulated with respect to the centre of mass.

We consider a system of n particles P_i (i = 1, 2, 3, ... n) having mass m_i are located in space with co-ordinates $\vec{r}_i$ (i = 1, 2, 3, ... n). The co-ordinate $\vec{R}$ of the centre of mass satisfies the condition



$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{R}\right) = 0 \qquad \text{.......... (1)}$$

or $$\sum_{i=1}^{n} m_i \vec{R} = \sum_{i=1}^{n} m_i \vec{r}_i$$

or $$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum_i m_i}$$

or $$\vec{R} = \frac{\sum m_i \vec{r}_i}{M} \qquad \text{.......... (2)}$$

Where $M = \sum_{i=1}^{n} m_i$ = mass of the system

If $\vec{r}_i'$ is the co-ordinate of the i-th particle with respect to the centre of mass from (1)

$$\sum_{i=1}^{n} m_i \vec{r}_i' = 0 \qquad (3)$$

Equation (2) or (3) may be taken as the definition of centre of mass.

Dynamics of a System of particles

We consider a system on n particles of mass m_i (i = 1, 2, ... n) at position vectors $\vec{r}_i$ (i = 1, 2, ... n) with respect to an arbitrary origin O.

If $\vec{F}_i$ denotes the total force on the i th particle, the equation of motion of the i th particle may be written as

$$m_i = \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i \tag{1}$$

The total force $\vec{F}_i$ on the i th particle consists of the external force on the i th particle $\vec{F}_i{}^e$ and forces of interaction of all other particles $\vec{F}_{1i}, \vec{F}_{2i}, ... \vec{F}_{ni}$; on the i th particle.

∴ We rewrite equation (1) as,

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^e + \sum_{j \neq i}^{n} \vec{F}_{ji}$$

Summing over all the particles we get.

$$\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_{j \neq i}^{n} \vec{F}_i^e + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \vec{F}_{ji} \tag{2}$$

The first term on the right hand side of eqn (2) is the total external force $\vec{F}^e = \sum_{n=1}^{n} F_i^e$ and the second terms contains each suffix twice and since $\vec{F}_{ji} = -\vec{F}_{ij}$ from third law of motion.

We get $\sum_{i=1}^{n} \sum_{j+i}^{n} \vec{F}_{ji} = 0$

We write eqn (2) as

$$\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}^e \tag{3}$$

If $\vec{R}$ is the position vector of the centre of mass and $\vec{r}_i'$ is the position vector of the i th particle with respect to the centre of mass

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

$$\frac{d^2\vec{r}_i}{dt^2} = \frac{d^2\vec{R}}{dt^2} + \frac{d^2\vec{r}_i'}{dt^2}$$

putting this in eqn (3) we get

$$\sum_{i=1}^{n} m_i \frac{d^2\vec{R}}{dt^2} + \sum m_i \frac{d^2\vec{r}_i'}{dt^2} = \vec{F}^e$$

or $$M\frac{d^2\vec{R}}{dt^2} = \vec{F}^e \qquad (4)$$

Since $$\sum_{i=1}^{n} m_i \frac{d^2\vec{r}_i'}{dt^2}$$

$$= \frac{d^2}{dt^2}\sum_{i=1}^{n} m_i\vec{r}_i' = 0.$$ From the defn. of centre of mass

Equation (4) states that the centre of mass moves as if the total external force were acting on the entire mass of the system concentrated at the centre of mass. Purely internal forces obeying third law of nation have no effect on the motion of the centre of mass.

The linear momentum of a system of n particles is

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i$$ $\vec{p}_i$ is the linear momentum of the i th particle.

$$= \sum_{i=1}^{n} m_i\dot{\vec{r}}_i = \frac{d}{dt}\sum_{i=1}^{n} m_i\left(\vec{R} + \vec{r}_i'\right)$$

$$= \sum_{i=1}^{n} m_i\frac{d\vec{R}}{dt} + \frac{d}{dt}\sum_{i} m_i\vec{r}_i'$$

or $\qquad \vec{p} = M\,\dot{\vec{R}} \qquad\qquad (5)$

Since $\sum_{i=1}^{n} m_i\, \vec{r}_i' = 0$ from the defn. of c.m.

The total linear momentum of a system of particles is the total mass of the system times the velocity of centre of mass.

$$\dot{\vec{p}} = \vec{F}_e$$

Rate of change of total momentum of a system is equal to total external force on the system. Therefore if $\vec{F}_e = 0$, $\vec{p}$ = constant vector. Total linear momentum of a system of particles is consurved if there is no external force on the system.

3.5 Total angular momentum of system

The total angular momentum of a system of *n* particles is given by.

$$\vec{L} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i$$

Since the angular momentum of the *i* th paricle is given by

$$\overrightarrow{Li} = \vec{r}_i \times \vec{p}_i$$

or
$$\vec{L} = \sum_{i=1}^{n} r_i \times m_i \dot{\vec{r}}_i$$

$$= \sum_{i=1}^{n} \left(\vec{R} + \vec{r}_i'\right) \times m_i \left(\dot{\vec{R}} + \dot{\vec{r}}_i\right)$$

$$= \sum_{i=1}^{n} \vec{R} \times m_i \dot{\vec{R}} + \sum_{i=1}^{n} \vec{R} \times m_i \dot{\vec{r}}_i + \sum_{i=1}^{n} \vec{r}_i' \times m_i \dot{\vec{R}} + \sum \vec{r}_i' \times m_i \dot{\vec{r}}_i$$

$$= \vec{R} \times \left(\sum_{i=1}^{n} m_i\right) \dot{\vec{R}} + \vec{R} \times \frac{d}{dt} \sum m_i \vec{r}_i' + \sum_{i=1}^{n} \left(m_i \vec{r}_i'\right) \times \dot{\vec{R}} + \sum_{i=1}^{n} \vec{r}_i' \times \vec{p}_i'$$

$$= \vec{R} \times M\dot{\vec{R}} + \sum_{i=1}^{n} \vec{L}_i'$$

Since the second and third terms contain $\sum m_i \vec{r}_i'$ which is zero from the defn. of c.m.

or $\qquad \vec{L} = \vec{L}_0 + \vec{L}_{cm}$

where $\qquad \vec{L}_0 = \vec{R} \times M\dot{\vec{R}}$, is the angular momentum of entire system concentrated at cm about the origin

$$= \vec{R} \times \vec{P}$$

$\vec{L}cm = \sum \vec{L}_i'$, is the angular momentum of system about the c.m.

Thus the angular momentum of a system of particles about an arbitrary origin is equal to the sum of the angular momentum of the entire mass concentrated at the centre of mass about the origin and the angular momentum of the system about the centre of mass.

3.6 Total Kinetic energy of a System of Particles

The total kinetic energy of a system of *n* particles is

$$T = \sum_{i=1}^{n} T_i = \sum \frac{1}{2} m_i v_i^2 \quad ,$$

where the kinetic energy of the *i* the particle is $T_i = \frac{1}{2} m_i v^2$

or, $$\mathrm{T} = \frac{1}{2} \sum_{i=1}^{n} m_i \; \vec{v}_i . \vec{v}_i = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\vec{v} + \vec{v}_i' \right) . \left(\vec{v} + \vec{v}_i' \right)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \left(m_i v^2 + m_i \vec{v} . \vec{v}_i' + m_i \; \vec{v}_i' . \vec{v} + m_i v_i'^2 \right)$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} m_i \right) v^2 + \vec{v} . \sum_{i=1}^{n} m_i \vec{v}_i' + \frac{1}{2} \sum m_i v_i'^2$$

$$= \frac{1}{2} m v^2 + \sum_{i=1}^{n} T_i' \qquad \sum m_i . \vec{v}_i' = \sum m_i \dot{\vec{r}}'$$

$$= \frac{d}{dt} \sum m_i \vec{r}_i' = 0$$

$\vec{v}_i$ = vel of the *i* th particle w.r.t. the orgin $\vec{v}$ = vel. of cm. w.r.t the origin $\vec{v}_i'$ = vel of i-th particle wrt cm.

and $\qquad T_i' = \frac{1}{2} m_i v_i^{-2}$

or $\qquad T = T_0 + T_{cm}\,; \qquad T_0 = \frac{1}{2} mv^2$ and $T_{cm} = \sum_i T_i'$

The kinetic energy of the system consists of two parts, the kinetic energy obtained if the entire mass is concentrated at the centre of mass (T_0) and the kinetic energy of the motion of the system about the centre of mass (T_{cm})

Conservation of energy

The work done by all the forces in moving the system from an initial configuration (1) to a final configuration (2) is

$$W_{12} = \sum_{i=1}^{n} \int_1^2 \vec{F}_i . d\vec{s}_i = \sum_{i=1}^{n} \int \left(\vec{F}_i e + \sum_{j+i}^{n} \vec{F}_{ji} \right) . d\vec{s}_i$$

$$= \sum_{i=1}^{n} \int \vec{F}_i^{\,e} . d\vec{s}_i + \int \sum_{i=1}^{n} \sum_{j \neq i}^{n} \vec{F}_{ji} . d\vec{s}_i$$

The Second term for say (ij) pair is

$$\vec{F}_{ji} . d\vec{s}_i + \vec{F}_{ij} \,.\, d\vec{s}_j$$

$$= \vec{F}_{ji} . \left(d\vec{s}_i - d\vec{s}_j \right) \qquad \because \; \vec{F}_{ij} = -\vec{F}_{ji}$$

$$= \vec{F}_{ji} \,. d\vec{s}_{ij} = 0\,, \quad \text{Since} \quad \vec{F}_{ji} \parallel d\vec{s}_{ji}$$

Hence we are left with

$$W_{12} = \sum \int_1^2 \vec{F}_i^{\,e} \,.\, d\vec{s}_i = \sum \int_1^2 m_i \dot{\vec{v}}_i \,.\, \vec{v}_i \; dt$$

$$= \sum_{n=1}^{n} \int_1^2 d \left(\frac{1}{2} m_i \; v_i^2 \right) = \sum_i \left(\frac{1}{2} m_i v_i^2 \right)_2 - \sum_i \left(\frac{1}{2} m_i \; v_i^2 \right)_1$$

$$T_2 - T_1 \qquad (1)$$

Again if the external force is derivable in terms of gradient of a potential (conservative force)

$$W_{12} = \sum \int \vec{F}_i e \, . \, d\vec{s}_i = -\sum \int_1^2 \left(\nabla_i V_i \, . \, d\vec{s}_i \right) = -\sum_{i=1}^{n} |V_i|_1^2$$

or $\quad W_{12} = V_1 - V_2 \qquad (2)$

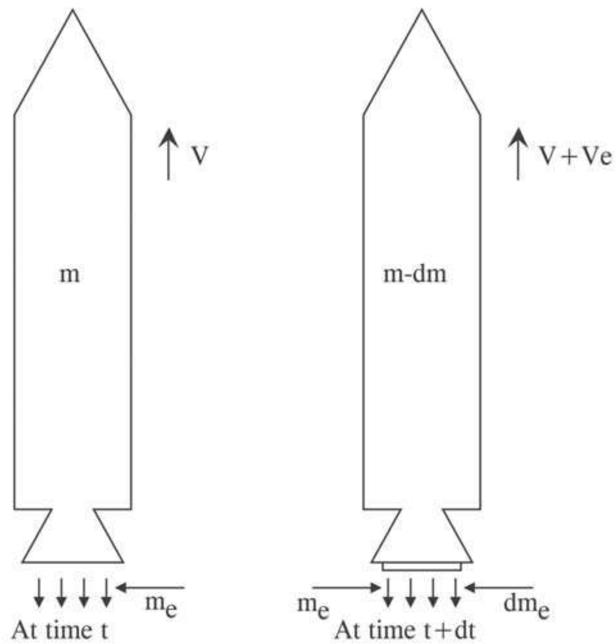
From (1) and (2)

$$T_1 + V_1 = T_2 + V_2$$

Total initial energy is equal to total final energy in case of conservative force field.

3.7 Rocket Motion

The rocket motion is based on the principle of conservation of momentum. Rocket is a device that can apply acceleration to itself using thrust by expelling a part of its mass with high velocity and can therefore move due to conservation of momentum. We establish the formula for rocket motion with the assumption that the motion is in vacuum, with no gravity and no air resistance. We consider two stages of motion of the rocket at two very close



consecutive time ‘t’ and ‘(t + dt)’. *m* is the mass of rocket and the propellant in it at the instant *t* and (m – dm_e) is that at time *t*+ dt. m_e and dm_e are the masses of rocket exhaust exited before *t* and during dt. *v* and *v* + *dv* velocity of the rocket at *t* and *t* + *dt*, J_e is the linear momentum of rocket exhaust at *t* and also at *t* + *dt* and v_e is the velocity of exhaust. All velocities are with respect to ground (inertial reference frame).

From principle of conservation of linear momentum we write

$$J_e + mv = J_e - dm_e V_e + (m - dm_e)\ (v + dv)$$

or $$O = m\ dv - (v + v_e)\ dm_e \quad \text{[neglecting the second order term dm dv]}$$

or $$(v+v_e)\frac{dm_e}{dt} = m\frac{dv}{dt}\,. \qquad (1)$$

The left hand side of above equation must represent the thrush acting on the rocket resulting the acceleration $\frac{dv}{dt}$. We write the thrush as -

$$T = (v+v_e)\frac{dm_e}{dt}\,.$$

or $$T = u\frac{dm_e}{dt}\,. \qquad (2)$$

$u = v + v_e$ is the velocity of exhaust relative to the rocket. This is approximetely constant in rockets. The term $\frac{dm_e}{dt}$ is the burn rate of rocket proplelant. We rewrite equation (1) as

$$dv = u\frac{dm_e}{m}\,. \qquad (3)$$

The mass of rocket exhausa dm_e is during the time interval dt is equal to the negative of the mass change of the rocket. Equation (3) becomes.

$$-u\frac{dm}{m} = dv$$

Integrating

$$\int_{v_i}^{v} dv = -u\int_{m_i}^{m}\frac{dm}{m}$$

or $\quad v - v_i = u \ln \frac{m_i}{m}$, m_i is the mass of the rocket with its contents and v_i is the velocity rocket at $t = 0$.

It we take $v_i = 0$

$$v = u \ln \frac{m_i}{m} \tag{4}$$

Equation (4) is an important equation in rocket motion known as Tsiolkovsky equation.

When rocket starts from ground it is necessary to include the force of gravity and the effect of air resistance. We then modify equation (1) as

$$u \frac{dm_e}{dt} - mg - F_D = m\frac{dv}{dt}$$; F_D is the atmospheric ohag.

or $\quad T - mg - F_D = ma$, a acceleration of rocket $= \frac{dv}{dt}$.

Problem :

(1) Calculate the mass ratio needed to escape earth's gravity starting from rest. It is given that escape velocity is 11.2×10^3 ms^{-1} and the exhaust velocity is 2.5×10^3 ms^{-1}.

Solution : We know that

$$v = u \ln \frac{m_i}{m}$$

or $\quad \ln \frac{m_i}{m} = \frac{v}{u} = \frac{11.2 \times 10^3}{2.5 \times 10^3} = 4.48$

$$\frac{m_i}{m} = e^{4.48} = 88$$

i.e. $\quad \frac{87}{88}$ part of final and $\frac{1}{88}$ part of rocket,

Problem :

(2) A rocket engine ejects at the rate of 40 kg s^{-1} with an exhaust velocity 4000 ms^{-1} with an exhaust velocity 4000 ms^{-1}. Rocket's initial mass is 24000 kg. What is the change of relocity of the rocket in 24 s?

Solution :

$$\frac{dme}{dt} = 40\ kg / s$$

fuel spent in 24s is 40 × 24 = 960 kg. using the rocket formula.

$$v = u \ln \frac{m_i}{m}$$

$$= 4000 \ln \frac{24000}{24000 - 960} = 4000 \times \ln 1.04$$

$$= 4000 \times .0408 = 163.2 \text{ ms}^{-1}.$$

Problem :

(3) In a rocket launching the initial mass was 2.8×10^6 kg. The fuel was found at the rate of $1.4 \times 10^{+4}$ kg s^{-1} to burn. The exhaust velocity was 2.4×10^3 ms^{-1} calculate the initial acceleration.

The thurst is

$$T = u \frac{dm_e}{dt} = 2.4 \times 10^3 \times 1.4 \times 10^4 \text{ kgm s}^{-2}$$

$$= 3.36 \times 10^7 \text{ kg ms}^{-2}$$

acceleration produced

$$= \frac{T}{m_e} = \frac{3.36 \times 10^7}{2.8 \times 10^6} = 12 \text{ ms}^{-2}$$

considering gravitation effect the net acceleration is

$$12 - 9.8 = 2.2 \text{ ms}^{-2}$$

3.8 Solved Problem

(1) A boy of mass 40 kg jumps from rest into a trolley of mass 80 kg, moving with velocity 10 ms^{-1}. What will be the velocity of the trolley after the boy jumped in ?

Solution : Total linear momentum before jump's 40 × 0 + 80 × 10 = 800 $kgms^{-1}$

If v is the velocity of the trolley and boy after the jump, the total linear momentum is

$$(80 + 40)\ v = 120\ v.$$

From the priniciple of conservation of linear momentum.

$$120\ v = 800$$

$$v = \frac{800}{120} = 6.67\ \ ms^{-1}$$

(2) A 5 kg gun fires a bullet of 15 gm at a velocity 1000 ms^{-1}. What will be the velocity of recoil of the gun.

Total linear momentum of the system of gun and bullet before fire is

$$50 \times 0 + \frac{15}{1000} \times 0 = 0\,.$$

If v is the velocity of recoil, the total linear momentum of the system is

$$50 \times v + 1000 \times \frac{15}{1000} = 5v + 15$$

$$\therefore \quad 5v + 15 = 0, \quad v = -3\ ms^{-1}.$$

The recoil velocity is 3 ms^{-1} oppositive to the direction of motion of the bullet.

(3) Two masses m_1 and m_2 move along a line, m_1 following m_2 with velocity v_1 while m_2 moves with a velocity v_2 ($v_1 > v_2$). They collide and stick with each other and continue moving along the same line with velocity $(v_1 + v_2)/2$. Find the relation between the masses.

Solution : Total linear momentum along the path before collission is

$$m_1v_1 + m_2v_2$$

Total linear momentum in the same direction after the collission is

$$(m_1 + m_2)\frac{(v_1 + v_2)}{2}$$

from convervation of linear momentum

$$m_1v_1 + m_2v_2 = (m_1 + m_2)\frac{(v_2 + v_1)}{2}$$

or $\quad 2m_1v_1 + 2m_2v_2 = m_1v_2 + m_1v_1 + m_2v_2 + m_2v_1$

or $\quad m_1(v_1 - v_2) - m_2(v_1 - v_2) = 0$

$$(m_1 - m_2)\ (v_1 - v_2) = 0$$

Since $\quad v_1 \neq v_2, \quad m_1 = m_2$

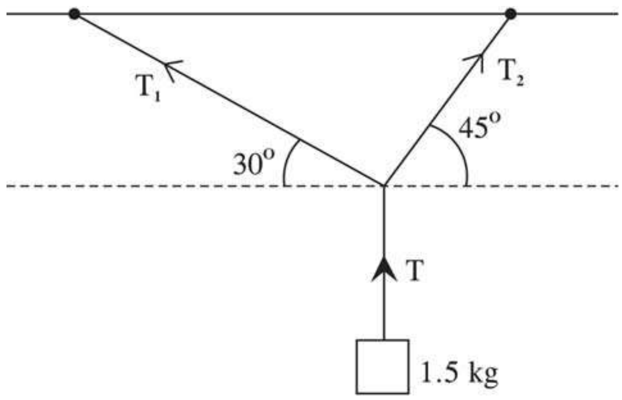
3.9 Exercises

(1) A 1.5 kg mass has an acceleration $\left(4\hat{i}-3\hat{j}\right)$ ms^{-2}. Two forces act on the mass. one is . What is the other force ?

(2) Two masses 4 kg and 8 kg at the ends of a rope are hung and pass through a frictionless pulley (At wood machine) calculate the acceleration of the masses and the tension of the rope.

(3) A 100 m long iron chain of mass 10 kg m^{-1} is being dragged by a force of 200 N. What will be the tension of the chain at its mid point ?

(4) A mass of 1.5 kg is hung as shown in the figure below. Compute the value of the tensions T_1 and T_2.



(5) How do Newton's Laws define and measure force ?

(6) How we derive Newton's first law and the measure of inertia from Newton's Second law ?

(7) Two different masses m_1 and m_2 moving directly towards each other with velocities v_1 and v_2 ($v_1 \neq v_2$). After collission they stick together and stops. Find the condition.

(8) On a smooth surface a ball A of mass 100 g at a velocity 10 ms^{-1} elastically collide with another ball B of mass 200 g moving with velocity 5 ms^{-1} in the same direction. They collide and keep on moving in the same direction. Compute their velocities after collission.

(9) A train car of mass 42000 kg moving with velocity 10 m/s yowardss another train car. After the two car collide, they couple together and move with velocity 6 ms^{-1}. What is the mass of the second train car ?

Unit 4 ❐ Rotational Motion

Structure

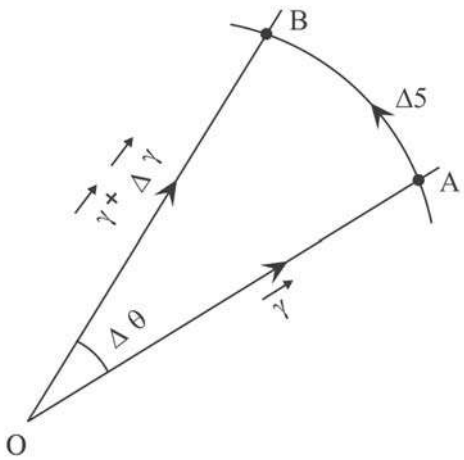
4.1 Objectives

A knowledge of circular motion & its application will develop by going through this unit.

4.2 Introductions

A point mass *m* moving along line AB as shown, seems to rotate about an arbitrary origin O. In small time Δt it moves from A to B with linear speed v. AB subtends an angle $\Delta\theta$ at *O*. The angular velocity of the point mass about O is defined as

$$\vec{\omega} = \underset{\Delta t \to 0}{Lt} \frac{\overrightarrow{\Delta\theta}}{\Delta t} = \underset{\Delta t \to 0}{Lt} \frac{\Delta\theta}{\Delta t} \hat{n}$$



$\overrightarrow{\Delta\theta}$ is a vector quantity given by $\overrightarrow{\Delta\theta} = \Delta\theta\,\hat{n}$

where $\hat{n}$ is a unit vector given by the direction of motion of a right hand screw head when rotation from A to B.

$$\Delta\theta = \frac{\Delta s}{r} = \frac{v\Delta t}{r}$$

$$\vec{\omega} = \hat{n} \underset{\Delta t \to 0}{Lt} \frac{\Delta\theta}{\Delta t} = \hat{n}\frac{v}{r}$$

$\omega = \frac{v}{r}$ is the magnitude of the angular velocity

4.3 Angular Momentum

Angular momentum is the moment of linear momentum of the point mass about the origin is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

$\vec{p} = m\vec{v}$, is the linear momentum

Angular momentum $\vec{L}$ is a vector quantity $\perp^r$ both to the radius vector $\vec{r}$ and linear momentum $\vec{p}$.

In magnitude (when r & v are perpendicular)

$$L = rmv = mr^2 \frac{v}{r}$$

$$= mr^2\omega = I\ \omega^2$$

$I = mr^2$ is known as moment of inertia of the point mass about the origin.

4.4 Kinetic Energy

The kinetic energy of point mass is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\frac{v^2}{r^2}$$

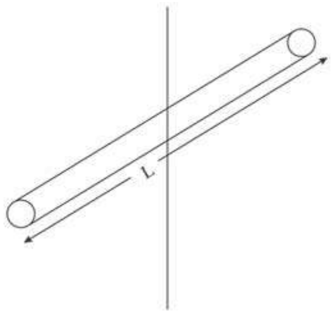
$$= \frac{1}{2}I\omega^2$$

In linear motion the linear momentum and kinetic energy are given by $p = mv$ and $T = \frac{1}{2}mv^2$ whereas corresponding physical quantities in rotational motion are angular momentum $L = I\omega$ and kinetic energy $T = \frac{1}{2}I\omega^2$. These similarity of expressions may be used to give a meaning to the new physical quantity moment of inertia.

Moment of inertia plays the same role in rotational motion as played by mass in linear motion. Mass is the inertia in linear motion and moment of inertia is the inertia in rotational motion.

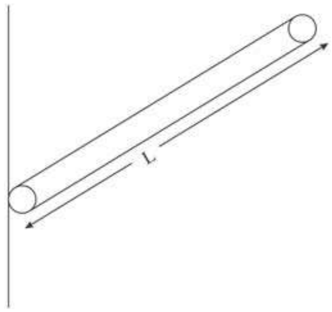
The moment of inertia of extended bodies with different shapes having different axes of rotation are of inmense importance in Physics and Engineering and can be calculated with the use of simple mathematics. A few are given below.

1. Thin rod of length 'L' and mass 'm'. Axis perpendicular to the rod and through the centre.



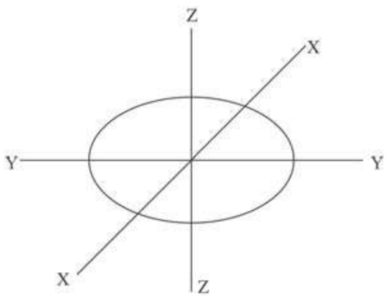
$$I = \frac{1}{12} mL^2$$

2. Thin rod of length L and mass 'm'. Axis perpendicular to the rod and through and end.



$$I = \frac{1}{3} mL^2$$

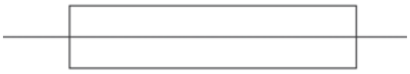
3. Thin solid disc of radius '*r*' and mass '*m*'. Axis $\perp^2$ to the disc through the centre and any axis in the plane of disc and through the centre (Through any diameter of the disc.



$$I_x = I_y = \frac{1}{4} m\gamma^2$$

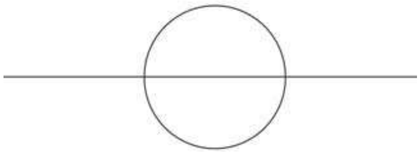
$$I_z = \frac{1}{2} m\gamma^2$$

4. Solid cylinder of mass *m* and raius '*r*' about the axis of the cylinder.



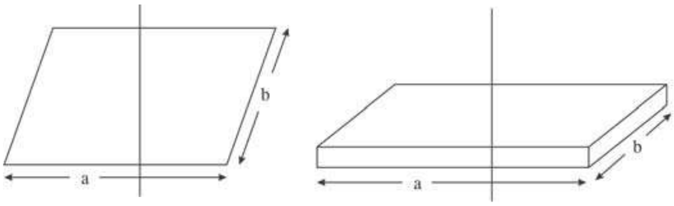
$$I = \frac{1}{2} m\gamma^2$$

5. Solid sphere of mass '*m*' and radius '*r*', about any diameter.



$$I = \frac{2}{5} mr^2$$

6. Thin rectangular plate or rod. Axis $\perp^r$ to the plane through the centre.



$$I = \frac{1}{12} . m\left(a^2 + b^2\right)$$

Torque and conservation of angular momentum.

We write Newton's Second law of motion as,

$$\vec{F} = \dot{\vec{p}} \qquad (1)$$

Taking the Cross product by $\vec{r}$ on both sides of equation (1) from left are have

$$\vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}} \qquad (2)$$

The angular momentum is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

The time rate of change of angular momentum is

$$\dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \qquad \dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times m\dot{\vec{r}} = 0$$

$$= \vec{r} \times \dot{\vec{p}}$$

The lefthand side of eqn (2) is the moment of force also known as torque ($\vec{N}$)

and right hand side is equal to the rate of change of angular momentum. We write eqn. (2) as

$$\vec{N} = \dot{\vec{L}}$$

in words the rate of change of angular momentum is equal to the torque applied on the point mass.

If the applied torque in zero the angular momentum will remain conserved.

In case of a system of (n) particles the total angular momentum is

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i$$

$\vec{L}_i$ is the angular momentum of the i th particle.

$$= \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i$$

The time rate of change of total angular momentum is

$$\dot{\vec{L}} = \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{p}}_i + \sum_{i=1}^{n} \dot{\vec{r}}_i \times \vec{p}_i$$

$$= \sum \vec{r}_i \times \left(\bar{F}_i^e + \sum_{j=1}^{n} F_{ji} \right)$$

$$\dot{\vec{r}}_i \times \vec{p}_i$$

$$= \dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i$$

$= 0$ for each i

or $$\dot{\vec{L}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^e + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \vec{r}_i \times \vec{F}_{ji} \qquad (3)$$

The last term of eqn (3) can be considered as sum of pairs

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = \left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{ji}$$

Since the mutual force between i th and j th particle is along the line joining them $\left(\vec{r}_i - \vec{r}_j\right) \parallel \vec{F}_{ji}$ hence each pain on the right hand side of eqn (3) is zero and equation (3) looks like

$$\dot{\vec{L}} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^e \qquad \vec{r}_i \times \vec{F}_i^e = \vec{N}_i^e \text{ is the external torque on the } i\text{th particle}$$

$$= \sum_{i=1}^{n} \vec{N}_i^e$$

or $\dot{\vec{L}} = \vec{N}^e \quad (4)$; $\vec{N}_e = \sum \vec{N}_i^e =$ total external torque on the system of particles.

In case $\vec{N}^e = 0$, $\vec{L} =$ const.

Equation (4) states that the time rate of change of angular momentum of a system of particles is equal to the total external torque on the system. In absence of external torque the angular momentum of the system will remain conserved.

4.5 Solved problem

1) If the earth had its radus suddenly decreased by half when spinning about its axis, what would be the length of the day ?

If L_1 and L_2 are the angular momentum of earth before and after decrease of the radius and I_1 and I_2 and ω_1 and ω_2 are respectively the moment of inertia and angular velocity in two cases ; from the conservation of angular momentum.

$$L_1 = L_2$$

or $$I_1\omega_1 = I_2\omega_2$$

or $$\frac{2}{5}MR_1^2\left(\frac{2\pi}{T_1}\right) = \frac{2}{5}MR_2^2\left(\frac{2\pi}{T_2}\right);$$

M = mass of earth R_1 and R_2 and radii and T_1 & T_2 are the period of revolution before & after.

or $$\frac{R_1^2}{T_1} = \frac{R_2^2}{T_2}$$

or $$T_2 = \left(\frac{R_2}{R_1}\right)^2 T_1$$

$$= \left(\frac{1}{2}\right)^2 \times 24 = 6 \text{ hrs.}$$

4.6 Exercises

(1) A ballet dancer spins about a vertical axis at 90 rpm with her arms outstretched. With her arms folded, the moment of inertia about the axis of rotation changes to 75%. Calculate the new rate of rotation.

(2) The mass and radius of earth are respectively 5.972×10^{24} kg and 6.378×10^{6} m. Compute the angular momentum of the earth.

(3) A disc of mass *M* and radius R is rotating about it axis with angular velocity ω, If a bob of clay of mass m is dropped on the rotating disc at distance $\frac{R}{2}$ from centre, calculate the resulting angular velocity.

Unit 5 ❒ Gravitation

Structure

5.1 Objectives

You will learn basic of Gravition & it application. Also you will gather knowledge abotu artificial satellite, GPS system etc.

5.2 Introduction

In the universe any two bodies attract each other with a force depending on their masses and distance between them.

Newton discovered a law known as Newton's law of gravitation which may be stated as a point particle of mass *M* attracts another point particle of mass *m* at a distance $\vec{r}$ from it with a force proportional to the product of the masses and inversely proportional to the square of the distance between them and the force is directed along the line joining them. Mathematically.

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\frac{\vec{r}}{r} = -\frac{k}{r^3}\vec{r}$$

$\vec{F}(\vec{r})$ is the force by *M* on *m*, $\vec{r}$ is the radius vector of *m* with rspect to the point mass *M*. The -ve sign ensures the force is attractive. k = GMm is a constant. G is a constant known as universal gravitational constant, universal in nature and is equal to 6.674×10^{-11} $\text{Nm}^{+2}\,\text{kg}^{-2}$. Such force which is a function of radius vector and directed towards a fixed point is known as central force.

The motion of planets about the star or those of satellites about the planets are examples of motion under central force, and the force between them are central forces.

Central motion is confined in a plane

Let $\vec{L}$ is angular momentum of body under central motion. It is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

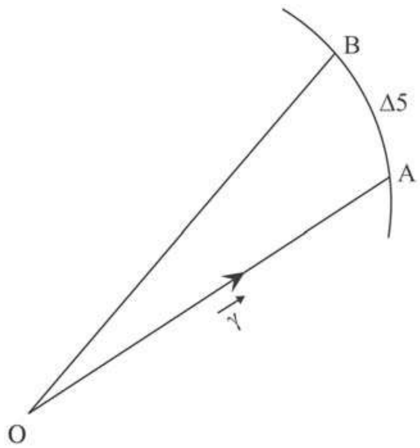
$$\dot{\vec{L}} = \vec{r} \times \dot{\vec{p}} + \dot{\vec{r}} \times \vec{p} \; ; \qquad \text{Since } \vec{p} = m\,\dot{\vec{r}} \qquad \dot{\vec{r}} \times \vec{p} = 0$$

or $$\dot{\vec{L}} = \vec{r} \times \vec{F}\,(\vec{r})$$

Since central for $\vec{F}\,(\vec{r})$ is parallel or antiparallel to $\vec{r}$ $\quad \vec{r} \times \vec{F} = 0$.

$\therefore$ In central motion $\dot{\vec{L}} = 0$ i.e. the angular momentum vector is constant. It is constant in magnitude as well in direction in space.

From definition of angular momentum $\vec{L} = \vec{r} \times \vec{p}$ the angular momentum vector is always $\perp^r$ to the plame containing $\vec{r}$ and $\vec{p}$ i.e $\dot{\vec{r}}$ and since direction of $\vec{L}$ is fixed $\vec{r}$ and $\vec{p}$ are always in a constant plane i.e. the motion is confined in a plane.



Let the body under central force has a mass *m* and instantaneous velocity $\vec{v}$. In time Δt it moves from *A* to *B*. The radius vector $\vec{r}$ sweeps an area AOB in time Δt.

The area AOB $= \frac{1}{2} r \Delta s$

or $$\Delta A = \frac{1}{2} r \cdot r \Delta\theta$$

The area swept by the radius vector per second is known as areal velocity and is given by

$$\underset{\Delta t \to 0}{Lt} \frac{\Delta A}{\Delta t} = \underset{\Delta t \to 0}{Lt} \frac{\frac{1}{2} r^2 \Delta\theta}{\Delta t} = \frac{1}{2} r^2 \underset{\Delta t}{Lt} \frac{\Delta\theta}{\Delta t}$$

$$= \frac{1}{2} r^2 \omega, \qquad \omega = \dot{\theta} = \underset{\Delta t \to 0}{Lt} \frac{\Delta\theta}{\Delta t} = \text{ angular velocity}$$

$$= \frac{1}{2} r^2 \frac{v}{r}$$

$$= \frac{1}{2} rv = \frac{mrv}{2m}$$

$$= \frac{L}{2m}, \qquad L = mrv = \text{magnitude of the angular momentum}$$

$$= \quad \text{Constant.}$$

This is a contant vector since the motion is confined in a plane.

Kepler's Laws

1) A planet moves in an elliptial orbit with sun at one of its foci.
2) A line segment joining a planet and the sun sweeps equal ara during equal interval of time.
3) The square of time period of revolution of a planet about the sun is proportional to the cube of the semi major axis of the elliptical orbit.

Artificial Satellites

Earth has a natural satellite, which move in an elliptic orbit. Man has placed artificial satellites in circular orbits around earth.

If a satellite of mass '*m*' orbits round the earth (mass M) in a circular orbit of radius *r* (the distance of the Satellite from earth's centre) with velocity *v*, the centrifugal force on the satellite is $\frac{mv^2}{r}$. The gravitational force of earth on the satellite is $\frac{GmM}{r^2}$. The satellite will remain in the orbit if

$$\frac{GmM}{r^2} = \frac{mv^2}{r}; \; v = \sqrt{\frac{GM}{r}}$$

Geostationary Satellite

A circular orbit in earth's equitorial plane of radius 4216 km and earth's centre as the centre is known as geostationary orbit. An object, an artificial satellite, in such orbit, known as geostationary satellite has an orbital period equal earth's rotation period of one sidereal day (23 hrs 56 min 4s). Such satellite looks motionless with respect to an observer on the surface of earth.

Geosyrchronous Satellite

Geosynchronous satellite is a satellite in geosynchronus orbit, with an orbital period same as earth's rotation priod (23 hrs 56 min 4 sec). Such a Satellite returns to the same position in the sky after each Sidereal day.

5.3 Application of Artificial Satellites

1. Weather forecasting : Various climate factors such as air pressure and temperature, humidity are monitored by using special cameras and instruments stationed in the artificial satellites. The records obtained are of immense importance for forcasting present and future weather.
2. Communication : Geostationary satellites are used for communication purpose such as long distance telephone, telex, radio, TV etc.
3. Spying – Satellites are used to keep an eye the enemy troops, their position, movement etc.
4. Study of the outer space.
5. Information about natural resources on earth such as underground water, minerals, oil wells, natural gas, coal deposits etc.

Artificial satellites are therefore of immense importance in research, defence, remote sensing, movement of fish in oceans, global positioning and navigation.

5.4 Global Positioning System (GPS)

Global positioning system is a Satellite navigation system that allows land sea and airbornes users to determine their location, velocity and time 24 hrs a day in all weather condition, anywhere in the world. It is a system of 31 satellites designed to help navigation. To GPS receivers are included in many commercial product such as automobiles, smart phones etc.

5.5 Astronaut's Health hazards

Human are well adopted to the physical conditions at the surface of earth. Journey into the space brings several changes in their body and mind. Some are temporary and some other are of critical concern. Common problems felt by astronauts in the initial condition of weightlessness is known as Space Adaptation Syndrone. It includes nausea, vomiting, vertigo, headache, lethergy etc. Long term weightlessness causes muscle atrophy, detercoration of skeleton known as space flight osteopenia, cardio vascular system functions are slowed down, decrease in the production of red blood cell, balance disorder of immune system is effected, loss of bodymass, nasal congestion, sleeplessness, flautulence, change of position and structure of brains, damage of gastrointestinal tissues and aging effect etc.

5.7 Solved Problem

(1) A 1500 kg satellite orbits earth at an altitude of 2.5×10^6 m. (i) What is the orbital speed of the satellite ? (ii) What is the period of rotation ? (iii) What is the kinetic energy of rotation ?

We know

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

or $$v^2 = \frac{GM}{R} = \frac{6.67\times10^{-11} + 5.96\times10^{24}}{6.4\times10^6 + 2.5\times10^6}$$

$$= \frac{6.67\times5.96\times10^{13}}{8.9\times10^6} = 4.466\times10^7$$

$$v = \sqrt{44.66}\times10^3 = 6682\ ms^{-1}$$

$$T = \frac{2\pi R}{T} = 5274S \qquad KE = \frac{1}{2}mv^2 = 4.32\times10^{10}\ J$$

(2) Two bodies of masses 5 kg and 6×10^{24} kg are placed with their centres 6.4×10^6 m apart. Calculate the force of attraction between two masses and initial accelerations.

Solution :

Force

$$F = \frac{GmM}{R^2} = \frac{5 \times 6 \times 10^{24}}{\left(6.4 \times 10^6\right)^2} \times 6.67 \times 10^{-11}$$

$$= 48.85 \text{ N}$$

Acceleration of 5 kg mass $= \frac{48.85}{5} = 9.77 \text{ ms}^{-2}$

Acceleration of 6×10^{24} kg mass $= \frac{48.85}{6 \times 24} = 8.142 \times 10^{-24} \text{ ms}^{-2}$

(3) An object, dropped on the surface of a planet falls 27 m in 3 s. The radius of the planet is 8×10^6 m. What is the mass of the planet ?

Solution :

The acceleration of the falling body is

$$g = \frac{2h}{t^2} = \frac{2 \times 27}{3 \times 3} = 6 \text{ ms}^{-1}$$

Now from Newton's law of gravitation

$$\frac{GmM}{R^2} = mg$$

$$\therefore \quad M = \frac{gR^2}{G} = \frac{6 \times 8 \times 10^6 \times 8 \times 10^6}{6.674 \times 10^{-11}}$$

$$= 5.75 \times 10^{24} \text{ kg.}$$

(4) What is the velocity of a low altitude satellite of earth ?

Solution :

From Law of gravitational force

$$\frac{mv^2}{(R+0)} = \frac{GmM}{(R+0)^2}$$

$$v^2 = \frac{Gm}{R} = \frac{6.67\times10^{-11}\times5.97\times10^{24}}{6.378\times10^6}$$

$$= 62.21 \times 10^6$$

m = mass of the Satellite, M = Mass of earth

R = Radius of earth

G = Gravitational Constant.

$$v = \sqrt{62.21}\times10^3 = 7887 \text{ ms}^{-1}$$

5.7 Exercises

(1) A 1000 kg Satellite is in synchronous orbit around earth. Calculate the i) orbital radius, ii) altitude of the satellite and iii) Kinetic energy of the satellite.

(2) A sphere of mass 100 kg is attracted by another sphere of mass 11.75 kg by a force of 19.6×10^{-7} N, when the distance between them is 0.2 m. Find the value of G.

(3) (i) If suddenly the gravitational attraction between earth and its satellite becomes zero what will happen to the satellite ?

(ii) Why is the weight of a body at the pole is more than that at the equater ?

(iii) The time period of a satellite is 5 hrs. If the separation between the earth and the satellite is increased to four times what will be the time period of the satellite ?

(4) The gravitational protential energy of a 500 kg satellite around a planet of mass 4.2×10^{23} is -4.8×10^9 J. Calculate the (i) orbital velocity of the satellite (ii) the kinetic energy and (iii) total energy of the satellite. [Ans. i) 3.098×10^3 m, (ii) 2.4×10^9 J (iii) -2.4×10^9 J]

Unit 6 ❐ Fluids : Surface Tension

Structure

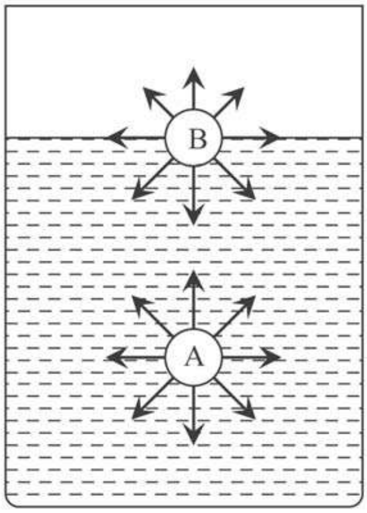
6.1 Objectives

A knowledge of fluid dynamis will be gatherd by you thogh this unit. The basics of surface tention & viscusity in discussed here.

6.2 Introductions

The free surface of a liquid behaves like a streched membrane and tries to minimise its surface area. The property of liquid surface by virtue of which it tries to minimise its surface area may be taken as qualitative definition of surface tension. A quantitative definition of surface tension can be obtained as follows. We imagine a line on the surface of a liquid. Both sides of line is pulled by a force due to contractile property of the liquid surface. The force of pull perpendicular to the line per unit length tangential to the surface is the measure of the surface tension. The unit of surface tension is Nm^{-1}.

Molecular Theory of Surface Tension

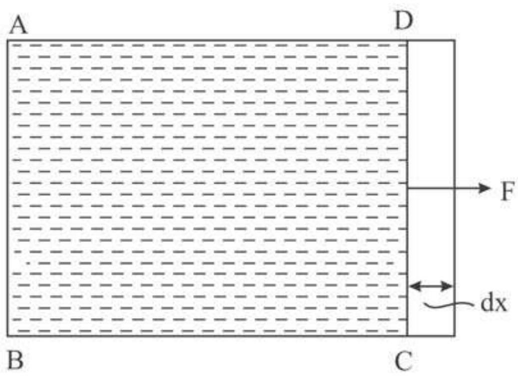


Let us consider some liquid in a container. The medium above the liquid is air say. A is a liquid molecule well inside the liquid. Liquid molecules symmetrically distributed about A pull A by short range molecular force. This force between molecules of same liquid is known as cohesive force. The net force on A is zero. B is a molecules on the liquid surface. From below B is pulled by liquid molecules and from above by air molecular. Molecules force between liquid molecules & air molecules is known as adhesive force. In this case cohesive force is stronger than the adhesive force. As a result the molecule B experiences a net downward force. Thus all molecules on the surface are pulled downwards. This unbalanced force results in a potential energy in the surface. Nature wants to minimise this potential energy by minimising the surface area and therefore the surface behaves as a stretched membrane.

6.3 Surface Energy

Due to unbalanced force on the surface molecules of a liquid the surface becomes a sheet of potential energy. If we take a soap film in a loop and pierce it with a pin we will see the liquids of the film to be blown around. This is a direct evidence of existence of surface enery.

To estimate the amount of energy on the liquid surface we take a rectangular liquid film formed in a rectangular frame ABCD. Whose CD can be moved. A force F = 2 × CD × T must be applied on CD to held the film. (The film has two surface and T is surface tension.) If the side CD is pulled further by dx the work by F is Fdx = 2. CD. Tdx. This work is stored in film as potential energy of the inereased area. If E stands for potential energy per unit area, the increase in surface energy will be 2E.CD.dx. Thus



$$2E.CD.dx = 2F.CD.dx$$

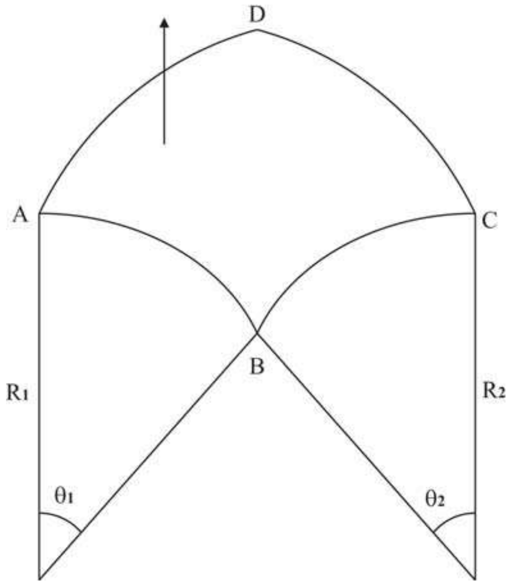
$$\therefore \quad E = T$$

The process is taken to be adiabatic.

The surface energy density is equal numerically to the surface tension having unit Jm^{-2}.

Excess pressure

Due to the property of surface tension a drop or bubble tries to contract and so compresses the matter enclosed. This in term increases the internal pressure which prevents further contraction and equilibrium is achieved. So in equilibrium pressure inside a bubble or a drop is greater than that outside and the difference of pressure is called excess pressure.



Expression of excess pressure

ABCD is an elementary portion of a curved liquid surface.

$$AB = CD = R_1\theta_1 \text{ and}$$

$$BC = AD = R_2\theta_2$$

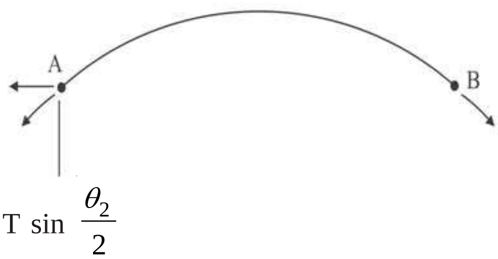
where R_1 and R_2 are respective radii of curvatures and θ_1 and θ_2 are the angles in radian subtended by AB and BC at the centre of their curvatures. Component of surface tension at A perpendicular to AB is T sin $\frac{\theta_2}{2}$ and that at B perpendicular to BC is T sin $\frac{\theta_1}{2}$.

The net inward force on AB is $2 \times T \sin\frac{\theta_2}{2} \times AB \; = \; 2T\frac{\theta_2}{2}R_1\theta_1 \quad [\theta_2 \text{ is small}]$

$$= TR_1\,\theta_1\,\theta_2$$

The net inward force on BC is $TR_2\,\theta_1\,\theta_2$

The excess force $\quad = TR_1\,\theta_1\,\theta_2 + TR_2\,\theta_1\,\theta_2$



$$= T\theta_1\,\theta_2\left(R_1 + R_2\right)$$

If P_1 and P_2 are the pressures inside and outside the curved surface

$$\left(P_1 - P_2\right) AB.\, BC = T\theta_1\,\theta_2\left(R_1 + R_2\right)$$

or, the excess pressure

$$\Delta P = P_1 - P_2 = \frac{T\theta_1\,\theta_2\left(R_1 + R_2\right)}{R_1\theta_1 \times R_2\theta_2}$$

$$= \frac{T\left(R_1 + R_2\right)}{R_1 \times R_2} = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

For spherical drop $R_1 = R_2 = R$ say

$$\Delta P = \frac{2T}{R}$$

For sphenical bubble having two liquid surfaces

$$\Delta P = \frac{4T}{R}$$

For cylindrical drop $R_1 = R,\ R_2 = \infty$

$$\Delta P = \frac{T}{R}$$

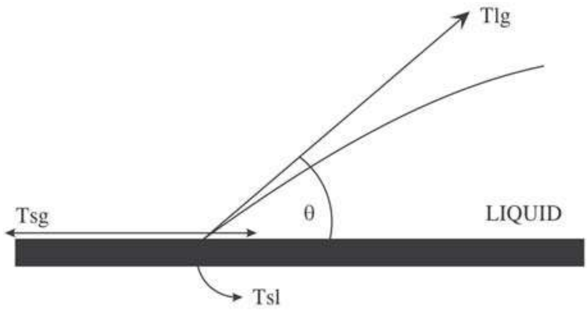
For cylindrical bubble

$$\Delta P = \frac{2T}{R}$$

6.4 Angle of contact

The angle conventionally measured through the liquid where liquid - vapour interface meets solid surface. It quantifies the wettability of a solid surface by a liquid. A given system of solid, liquid and vapour at a given temperature and pressure has a unique contact angle.

A liquid drop on a solid surface in a gaseous medium is shown in the figure. T_{lg} is the surface tension of the liquid in the gas medium acts tangentially on the liquid surface at the meeting point of solid liquid and gas. T_{sg} and T_{sl} are similar terms like surface tension for solid – gas and solid liquid pain. T_{sg}, better to be called solid surface free energy and T_{sl} solid-liquid interfacial free energy act as shown in the figure.



The young -Laplace equation for equilibrium is

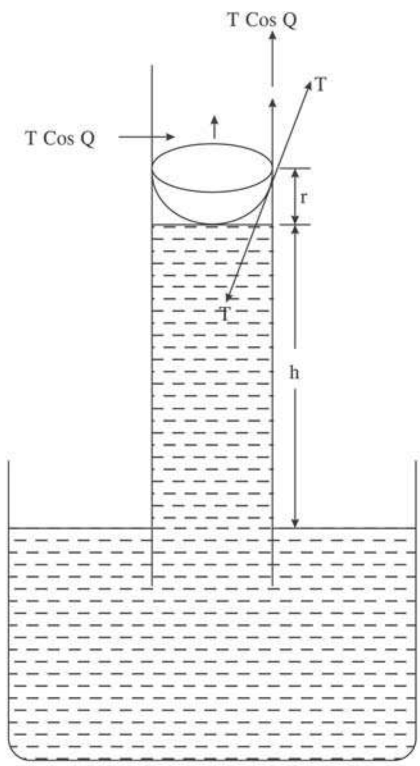
$$T_{sg} = T_{sl} + T_{\text{lg}} \cos\theta$$

or $$\cos\theta = \frac{T_{sg} - T_{sl}}{T_{\text{lg}}}$$

If $T_{sg} > T_{sl}$; θ is acute, the liquid is said to wet the solid surface as in the case of water on glass surface. In case $T_{sg} < T_{sl}$; θ is obtuse liquid does not wet the surface as in the case of mercury on glass surface.

Capillary rise

A Capillary tube is vertically immersed in a liquid in a container. If the liquid wets the surface, liquid will rise in the tube. The free surface within the tube is spherical. The liquid touches tube along a circle of radius *r* equal to the radius of the capillary tube. At the point of contact the surface tension *T* will act making an angle θ with the tube surface down-wards, where θ is the angle of contact. The reaction of the surface tension has a vertically upward component $T \cos\theta$, everywhere on the circle of contact. Hence the liquid column in the capillary tube will be pulled up by a force $T \cos\theta \times 2\pi r$. At equilibrium this force will be



equal to the weight of the liquid column of height h. The net volume of liquid column is $\pi r^2(h+r)-\frac{1}{2}\times\frac{4}{3}\pi r^3=\pi r^2\left(h+\frac{r}{3}\right)$. If d is the density of the liquid

$$T\cos\theta\times 2\pi r=\pi r^2\left(h+\frac{r}{3}\right)dg$$

or $$T\cos\theta=\frac{1}{2}\left(h+\frac{r}{3}\right)rdg$$

or $$h=\frac{2T\cos\theta}{dgr}\cdot \quad (1) \quad \text{Since } r<<h$$

or $$h=\frac{2T}{dgr}. \qquad (2) \text{ for small } \theta.$$

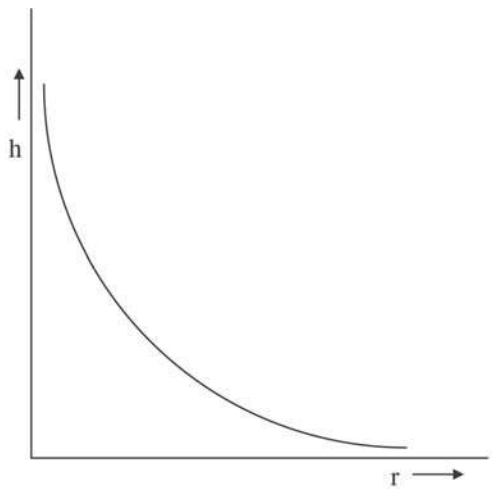
from equation (1) or (2)

for a particular liquid tube pair

hr = constant

This is known as Jurin's law.

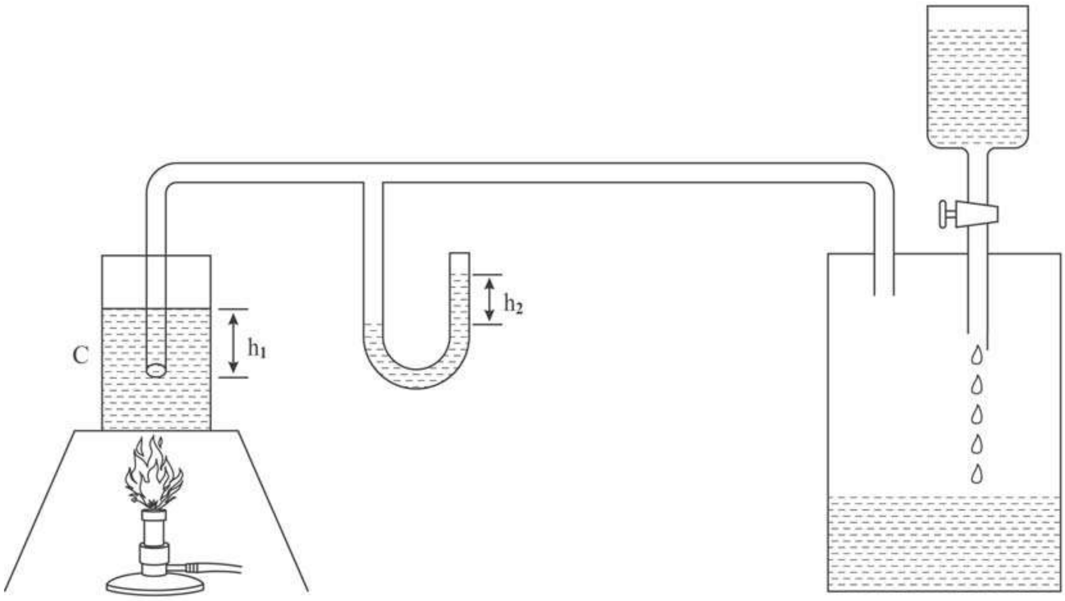
By measuring the height of the liquid column and radius of the capillary tube surface tension of liquid may estimated using equation (2).



6.5 Jaeger's method

In Jaeger's method surface tension of a liquid is determined by measuring the pressure required to cause air to flow from a capillary tube immersed in the liquid. One end of a capillary tube is vertically immersed in the experimental liquid kept in a container C. From other end air is pushed into the tube. The pressure inside is measured with the help of a momometer M.

Bubbles are formed at the tip of the capillary tube. It is assumed that the radius of bubble formed is equal to the radius of the capillary tube.



The excess pressure inside the bubble is $\frac{2T}{r}$, where T is the surface tension of the liquid to be measured and r is the radius of the capillary tube, which can be measured using a travelling microscope. If h_2 be maximum difference in liquid heights in the manometer, then the air pressure inside the bubble when it is just on the point of being detached from the orifice is $h_2 \rho_2 g + p$. Where p is the atmospheric pressure and ρ_2 is the density of manomatric liquid. The pressure outside the bubble is $h_1 \rho_1 g + p$, where ρ_1 is the density of the experimental liquid. Thus the pressure difference is

$$\frac{2T}{r} = \left(P + h_2\rho_2 g\right) - \left(P + h_1\rho_1 g\right)$$

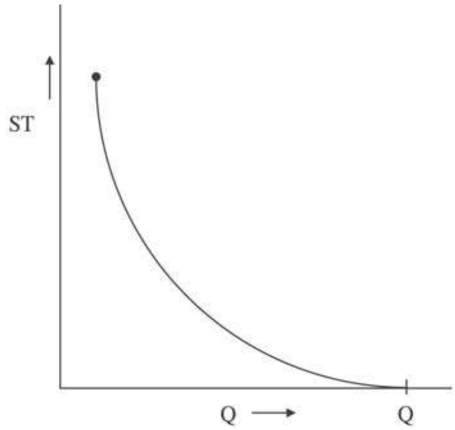
or $$T = \frac{rg}{2}\left(h_2\rho_2 - h_1\rho_1\right)$$

Jaeger's method is very useful for the study of variation of surface tension with temperature, comparison of surface tension of different liquids, variation of surface tension of solution with concentration of solute, determination of surface tension of molten metals etc.

Synclastic Surface : The curved surfaces having centres of curvatures on the same side of the surface such as dome shaped surface.

Anticlastic Surface : The curved surfaces having centres of curvatures on the opposite sides of the surface such as saddle shaped surface.

Variation of surface tension with temperature : –



For small ranges of temperatures the surface tension varies linearly with temperature, as

$T_t = T_o (1 - at)$ (T_t and T_0 are surface tension at temperature t and at $0^o c$, a is known as temperature coefficient of surface tension)

However the most comprehensive relation given by Eotuos may be written as $TV^{2/3} = k(\theta_c - \theta)$; V is the molar volume

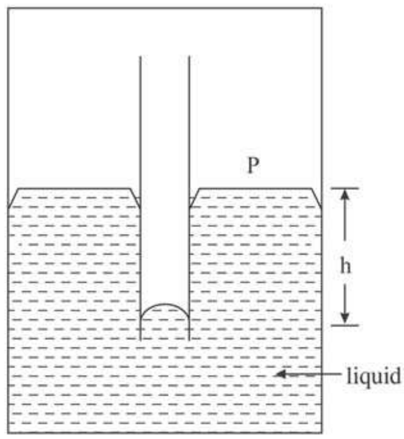
θ_c = Critical temperature where $T = 0$

θ = Temperature in Kelvin.

For water surface tension varies from 58.8 dyn cm^{-1} to 75.6 dyn cm^{-1} for temperature between 100^{o}c to 0^{o}c.

Surface tension & condensation of vapour :

We consider a narrow verticle glass tube, dipped into a liquid which does not wet the glass. The entire assembly is enclosed in air tight system as shown in the figure. If P is the saturated vapom pressure on the plane surface that on the convex surface is $P + h\sigma g$. where σ = density of saturated vapour and h is the depression liquid column in the tube. The pressure just inside the meniscus is $P + h\rho g$, where ρ is the density of the liquid. Thus the excess pressure is



$$\frac{2T}{r} = (P + h\rho g) - (P + h\sigma g)$$

or $$\frac{2T}{r} = (\rho - \sigma)hg > 0$$

Thus saturation vapour pressure on convex surface is greater than that on the plane surface. Similarly it can be shown that saturation vapour pressure on concave surface is less than that on plane surface.

If we place a drop of water in a space where vapour pressure is at saturation value for a plane surface, the drop will begin to evaporate for the vapour pressure is less than the saturation vapour pressure for the drop and it will therefore be converted into vapour in order to increase the vapour pressure to its own saturation value.

This will result in further decrease in radius of the drop and consequent rise in the saturation value of its own vapour pressure and it will therefore evaporate more and more rapidly. That is why saturated vepour does condense into drop, for as soon as a tiny drop is formed it begins to evaporate. Thus condensation may not take place even when the vapour becomes super saturated. If however dust particles or charged ions be introduced into the saturted vapour they offer flatter surface to it and condensation starts.

Phenomena associated with surface tension :

1) An immersed glass rod is taken out of the water, drops of water are seen sticking at the end of the rod.
2) Spider walks on the surface of water without rapturing it.
3) Blade or neddle left gently on water surface remains floating.
4) Paint brushes when taken out of paint the bristles are drawn closer.
5) Rain drops are spherical.
6) Irregular shaped camphor piece dances on water surface. Water surface in contact with camphor has lower surface tension, camphor moves to other side of higher surface tension.
7) Hot water having less surface tension is better than cold water for washing purpose
8) Detergent works on the principle of lower surface tension of solution.

Problem :

One thousand drops of water each of diameter 0.2 mm combine to form a single drop. Calculate the loss of energy. T = 72 dyn. cm^{-2}

Volume of 1000 drops = volume of single combined drop of radius r say

$$\therefore \quad \frac{4}{3}\pi\,(0.01)^3 \times 1000 = \frac{4}{3}\pi\, r^3$$

$$r^3 = (0.01)^3 \times (10)^3,$$

$$\therefore \quad r = 0.01 \times 10 = 0.1 \text{ cm.}$$

Total energy of 1000 small drops.

$$= 1000 \times 4\pi\,(0.01)^2 \times 72$$

Total energy of the large drop is

$$= 4\ \pi\ (0.1)^2\ \times\ 72$$

Loss of energy = $4\pi \times 72 \left\{1000 \times (0.01)^2 - (0.1)^2\right\}$ erg

$$= 4\pi\left(0.1 - 0.01\right) \times 72 = 82.79 \text{ erg}$$

6.6 Solved problems - I

(1) n droplets of equal size of radius 'r' coalesce to form a bigger drop of radius 'R'. If T is the surface tension of the liquid, then show that the energy liberated is $4\pi r^2 T\left(n - n^{2/3}\right)$.

Solution

The volume remains constant, therefore

$$n \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi r^3 ; \qquad R = r\, n^{2/3}$$

$$R^2 = r^2\, n^{2/3}$$

decrease in surface area

$$= n \times 4\pi r^2 - 4\pi R^2 = 4\pi r^2\left(n - n^{2/3}\right)$$

$\therefore$ the energy liberated $= 4\pi r^2 T\left(n - n^{2/3}\right)$

(2) Surface tension of water of 0^oc is 70 dyn cm^{-1}. Find the surface tension of water at 25^0c . [α = 0.027]

Solution

$$T = T_0(1 - \alpha t) = 70\ (1 - 0.0027 \times 25)$$

$$= 65.28 \text{ dyn cm}^{-1}$$

(3) A glass tube of internal diameter 3.5 cm and thickness 0.5 cm held vertically with

its lower end immensed in water. Find the downward pull on the tube due to surface tension. (Surface tension of water = 0.074 N/m)

Solution :

Internal diamen = 3.5 cm ; internal radius = 1.75 cm

External radius = 1.75 + 0.5 = 2.25 cm

Length along which the water and glass surface meet

$$= 2\pi\ (1.75 + 2.25) \times 10^{-2}\ \text{m}$$

$$= 8\pi \times 10^{-2}\ \text{m}$$

$\therefore$ Downward pull $= 8\pi \times 10^{-2} \times 0.074$ N.

$$= 0.0186\ \text{N}$$

(4) A niddle 10 cm long can just rest on the surface of water. Find the weight of the needle. (S.T. = 0.07 Nm^{-1})

Solution :

Water wets the heedle from two sides along the length, hence effective length is 10 × 2 = 20 cm.

The force of surface tension on the needle is equal to the weight of the needle.

Hence the weight of the needle

$$= 20 \times \frac{1}{100} \times 0.07\ N = 14 \times 10^{-3} = 0.014\ \text{N.}$$

(5) A reetangular plate 16 cm × 10 cm rests with its face on surface of water of surface tension 0.07 Nm^{-1}. Calculate the force required to pull the plate out of water.

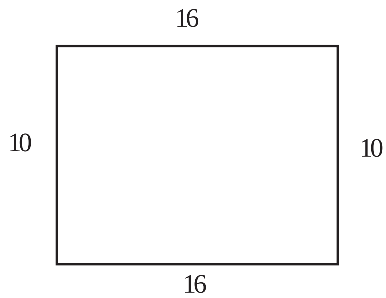
Solution :

The perimeter

$$= 2 \times (16 + 10) \frac{1}{100}\ \text{m}$$

$$= 52 \times 10^{-2}\ \text{m}$$

$\therefore$ The required force



$= 52 \times 10^{-2} \times 0.07 = 364 \times 10^{-4}$ N

$= 0.0364$ N

6.7 Viscosity

Viscosity is the property by virtue of which there is a resisting force against relative motion between two layes of a fluid. This property is reciprocal to fluidity and may be taken as the qualitative definition of viscosity. Quantitatively it is the force per unit area between two layers of fluid in contact per unit velocity gradient. Mathematically we write

$F \propto A$ F = resisting force

$\propto \dfrac{dv}{dx}$ A = area of layer in contact.

or $F \propto A\dfrac{dv}{dx}$ $\dfrac{dv}{dx}$ = velocity gradient

or $F = \eta\, A\dfrac{dv}{dx}$ η = coefficient of viscosity

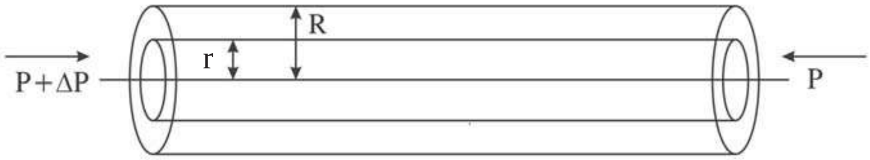
$$\eta = FA^{-1}\left(\frac{dv}{dx}\right)^{-1}$$

The unit of coefficient of viscosity is $Nm^{-2}s$. This is also known as Pascal second or Pas. Though Pas is the S1 unit of coefficient of viscosity it is not very popular in science and technology. Instead dynes per square centemeter dyns cm^{-2}, known as poise [P] is in use.

1 Pa s = 10 P

6.8 Poiseulle's law

Poseuille's law is a physical law that gives the pressure drop in an incompressible and Newtonian fluid in Laminar fluid through a long cylindrical tube of constant crossction. Laminar flow in the cylindrical tube prescribes that bunch of circular of layers fluid each having velocity determined by their radial distance from the axis of the tube – centre moving fastest while liquid in contact of the tube wall is stationary.



Let L and R are the length and radius of the capillary tube through which the fluid is passing. A concentric solid cylinder of the fluid of radius r is considered. If ΔP is the pressure difference at the two ends of the tube the force on the liquid cylinder pushing the liquid out is $\Delta P \pi r^2$. The viscous force against the motion of the cylindrical fluid is $\eta \, 2\pi \, rL \frac{dv}{dr}$, where $\frac{dv}{dr}$ is the velocity gradient.

For no acceleration

$$\eta \, 2\pi \, rL \frac{dv}{dr} = \Delta P \, \pi r^2$$

or
$$dv = \frac{\Delta P}{2\eta L} rdr \, . \tag{1}$$

Value of fluid velocity at axis is maximum, at radius r is v and at radius R is zero. Integration equation (1) we get

$$\int_0^v dv = \frac{\Delta P}{2\eta L} \int_R^r r \, dr$$

or
$$v(r) = \frac{\Delta P}{4\eta L}\left(R^2 - r^2\right) \tag{2}$$

Next we consider a cylindrical cell of the fluid between ra g out of the capillary tube with fluid of volume dQ per see is given

$$dQ = 2\pi r \, dr \, v = 2\pi \, \frac{\Delta P}{4\eta L} r\left(R^2 - r^2\right) dr \, .$$

or
$$dQ = \frac{\pi \Delta P}{2\eta L}\left(R^2 - r^2\right) rdr$$

Total fluid coming out of tube per sec is

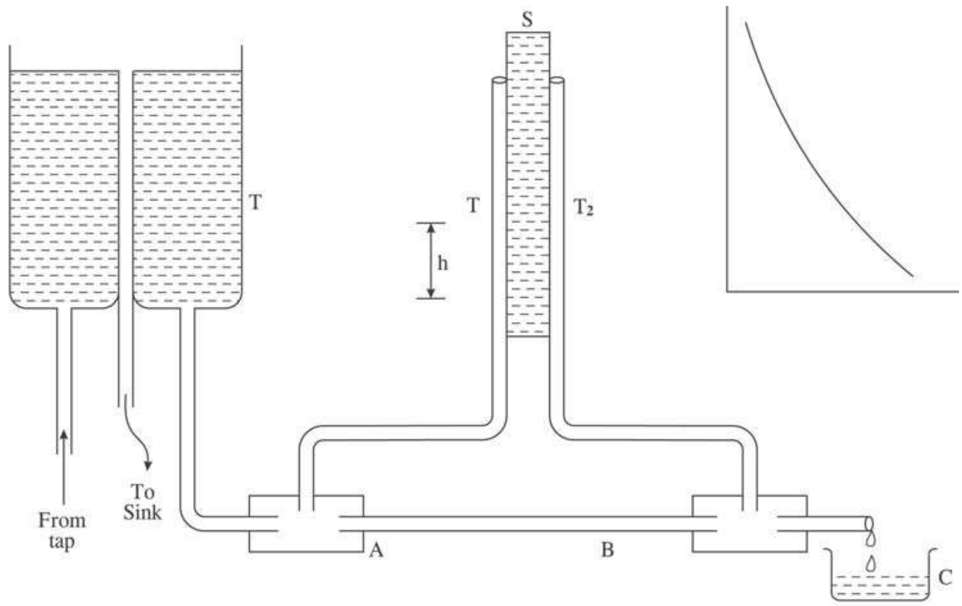
$$Q = \frac{\pi \Delta P}{2\eta L} \int_0^R \left(R^2 - r^2\right) rdr \, .$$

$$= \frac{\pi \Delta P}{2\eta L}\left(\frac{R^4}{2} - \frac{R^4}{4}\right)$$

or $$Q = \frac{\pi \Delta P \ R^4}{8 \ \eta \ L} \qquad (3)$$

Poiscuilles method of 'experimental determination of viscosity of liquid'.

In the diagram the experimental arrangement for the determination of viscosity of liquid is shown. AB is a long capillary tube. Experimental liquid from a constant level tank *T* enters the capillary tube from the end *A*. From the end *B* the outcoming fluid is collected



in the container C. In the manometric arrangement two vertical tubes T_1 and T_2 in connection with the end A and B respectively show different heights in manometric liquid. The height difference is gives the pressure difference Δp at the two ends A and B.

$\Delta P = hdg$, d = density of manometric liquid

The volume V of liquid collected in the container *C* in time *t* gives the rate of volume collected *Q* per unit time.

$$Q = \frac{V}{t} \ .$$

Using the Poiscuille's formula

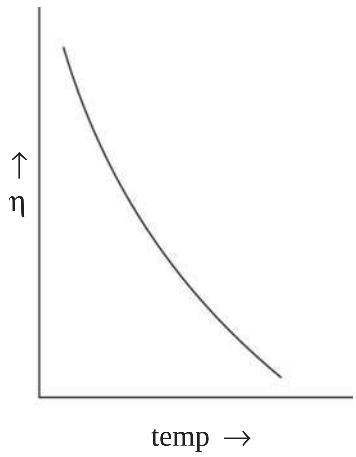
$$Q = \frac{\pi\, \Delta P}{8\eta L} R^4$$

The coefficient of viscosity (η) is estimated from

$$\eta = \frac{\pi\, \Delta P R^4}{8\, QL}$$

R is measured by the help of travelling microscope.

Variation of viscosity with temperature. In liquid due to strong cohesive forces between the molecules in any layer in moving liquid tries to drag the adjascent layer and produce the effect of viscosity. Increase in temperature breaks the bonding between the atoms and cohesive forces decreases. As a result the viscosity decreases. In gases where cohesion is less, the source of viscosity is the collision between the molecules. Higher the temperature more frequent the collision greater is the value of coefficient of viscosity. In these cases the cofficient of viscosity is proportional to square root of absolute temperature.



6.9 Lubrication

Lubrication is the action of applying a substance such as oil or grease to an engine or component as to minimise friction and allow smooth movement. Lubricant is a substance introduced to reduce friction between surfaces in mutual contact. Lubricity, the property of reducing friction must have

(1) High boiling point and low freezing point.

(2) High viscosity index

(3) Thermal Stability

(4) Hydrolic stability

(5) Ability of corrosion prevention and

(6) High resistance to oxidation.

If the lubicant is too thick it will require large amount of energy to move and if it is too thin the surfaces will come in contact and friction will increase.

6.10 Reynold's number

If fluid flows through a tube with small velocity the flow is steady. As velocity is gradually increased at one stage the flow becomes turbulent. The largest velocity that allows a steady flow is called critical velocity.

Reynold showed that the critical velocity v_c of a fluid in a tube is

i) $v_c \propto \frac{1}{\rho}$, ρ desnsity of the fluid

iii) $v_c \propto \eta$, η = coefficient of viscosity of the fluid.

iii) $v_c \propto \frac{1}{r}$, r = radius of the tube.

Thus $v_c = k\frac{\eta}{\rho r}$; k = constant of proportionality known as Reynold's number. If k < 2000, the flow is steady.

Example :

The velocity of water in a river is 20 km h^{-1} near the surface. If the river is 10 m deep, find the shear between the horizontal layers of the water of the river η = 10^{-2} poise.

We know $\frac{F}{A} = \eta\frac{dv}{dx}$ is the shear.

In this case $\frac{dv}{dx} = \frac{20\times10^3}{60\times60\times10} = 0.56$ s^{-1}

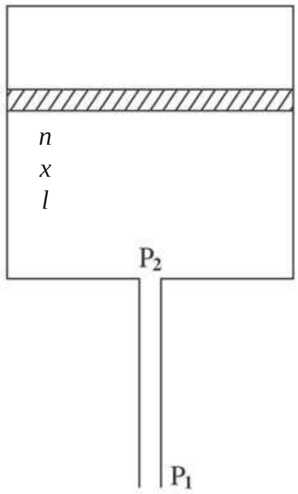
$$\frac{F}{A} = 10^{-2}\times0.1\times.56 = 5.6\times10^{-4} \text{ N m}^{-2}$$

6.11 Solved problems - II

(i) A tank 100 cm × 100 cm × 100 cm is full of water. Water is coming out through a vertical tube of length 40 cm and radius 1 mm from the bottom of the tank. If the coefficient of viscosity of water be 0.01 poise how much time will be required to loose half the tank of water ?

Solution :

From the Poiseuille's law the rate of flow is



$$Q = \frac{(P_2 - P_1)\,\pi r^4}{8\eta l},$$

$$Q = -\frac{100}{100} \times \frac{100}{100} \times \frac{dx}{dt}$$, when the height of water is x in the tank.

$$Q = -\frac{dx}{dt}.$$

The pressure difference $P_2 - P_1 = xg$

$$\therefore\ -\frac{dx}{dt} = \frac{x\,g\pi\,r^4}{8\,\eta\,l}.$$

or $$-\int_{1.00}^{0.50} \frac{dx}{x} = \int_0^t \frac{g\pi\,r^4}{8\,\eta\,l}\,dt$$

or $$\ln\frac{1}{0.5} = \frac{9.8 \times 3.14 \times \left(10^{-3}\right)^4}{8 \times \frac{0.01}{10} \times \frac{40}{100}}\,t$$

or $$0.693 = \frac{9.8 \times 3.14 \times 10^{-12}}{8 \times 4 \times 10^{-4}}\,t.$$

$$t = \frac{8 \times 4 \times 0.693}{9.8 \times 3.14} \times 10^8 = 7.2 \times 10^7\,s.$$

(2) The flow rate of blood in a coronary artery of a man is reduced to half its normal value by plaque deposits. By what factor has the radius of the artery been reduced ?

Solution

From Poiseuille's law

$$Q = \frac{(P_2 - P_1)\pi r^4}{8\eta l}$$

We write

$$\frac{Q_1}{r_1^4} = \frac{Q_2}{r_2^4} \quad \text{or} \quad \frac{Q_1}{Q_2} = \left(\frac{r_1}{r_2}\right)^4$$

or $$\left(\frac{r_1}{r_2}\right)^4 = \frac{Q_1}{Q_2} = \frac{Q_1}{Q_{1/2}} = 2$$

$$\frac{r_1}{r_2} = (2)^{\frac{1}{4}} = 1.189$$

$$r_2 = \frac{r_1}{1.189} = 0.84\, r_1$$

Thus the decrease in the artery radius is 16%.

(3) Two horizontal plates 250 mm apart have oil of viscosity 20 poise in between. Calculate the shear stress in oil if the upper plate is moved with velocity 1250 mm s^{-1}.

Stress = viscosity × velocity gradient

$$= \frac{20}{10} \times \frac{1250 \times .0^{-3}}{250 \times 10^{-3}}$$

$$= 19 \ NM^{-2}$$

6.12 Exercises

(1) A water film is formed between two straight parallel wires of length 10 cm each with a separation of 0.1 cm. If the distance between the wires is increased by 0.1 cm, how much work is to be done ? (S. T. of water = 0.072 N m^{-1})

(2) The radii of the two vertical arms of a u-tube are r_1 and r_2. When a liquid of density ρ and angle of contact zero is taken in this U-tube the difference of liquid in two arms is h. Show that the surface tension of the liquid is

$$\frac{h\rho g\, r_1\, r_2}{2(r_2 - r_1)}.$$

(3) Find the workdone in blowing a soap bubble of radius 10 cm. The surface tension of soap solution is 30 dyn cm^{-1}.

(4) A thin wire is bent in the from of a rectangle of dimension 5 cm × 4 cm. What force due to surface tension does the sides experience when soap film is formed in the frame ? (Surface tension of soap solution is 0.03 Nm^{-1})

(5) Mercury has an angle of contact 120^o with glass. A narrow tube of radius 2 mm made of this glass is dipped in a trough containing mercury. Calculate the capillary descent. S. T. of mercury = 0.456 Nm^{-1}. Density of mercury = 13.6×10^3 Kgm^{-3}.

(6) A plate 10 cm × 10 cm is pulled with a velocity 0.05 ms^{-1} in a liquid of viscosity 1.0 poise above a fixed plate at a distance 0.25 mm. Find the force to maintain the velocity.

(7) A plate in water 0.05 mm above a fixed plate is moving with velocity 100 cm s^{-1}. It requires a force at 2 Nm^{-2} to maintain the speed. Determine the viscosity.

Unit 7 ❐ Elasticity

Structure

7.1 Objectives

The main objective is to give you an idea about Elasticity of solid material and its different application in daily life.

7.2 Introduction

Elasticity is a property of matter by virtue of which it resists any external force trying to deform its shape and size (and the body regains its original shape and size when the external force is withdrawn)

Iron is much more difficult to be stretched than rubber hence iron has greater elasticity than rubber.

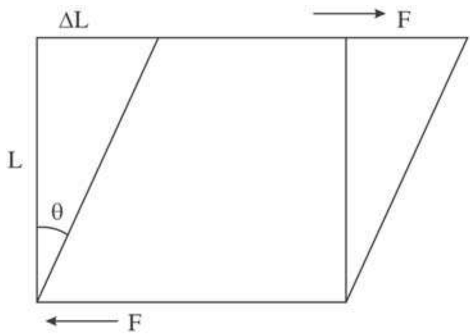
Strain – The fractional deformation of a body is known as strain.

Longitudinal strain – suppose a rod of length L is linerly deformed by an amount ΔL (i.e. it is alongated to $L + \Delta L$ or contracted to $L - \Delta L$) then the fractional deformation i.e. $\left(\Delta L/L\right)$ is the longitudinal strain. It is a dimensionless quantity.

Volume strain – Suppose a body of volume V is deformed by ΔV the fractional volume change $\left(\Delta V/V\right)$ is known as the volume strain.

Shearing Strain or Shear :–

We consider a cube of each side equal to L. on the upperside a tangential force F is applied. Due to friction an equal and opposite force F will be applied on the lower surface tangentially. The shape of the cube will be deformed. Let the upper side moves by an amount ΔL along the force. The fractional deformation $\Delta L/L = \theta$ is known as shearing strain or shear.



Whenever a body is strained a resisting force is generated within the body to oppose the deformation. This resisting force generated measured per unit area of application is called stress. Corresponding to above three type of strains we have longitudinal stress, volume stress and shearing stess. The unit of stress is Nm^{-2}.

Strain is the cause and stress is the result.

Hookes Law

Within elastic limit the stress is proportional to strain. i.e.

$$\frac{\text{Stress}}{\text{Strain}} = \text{constant}$$

This constant is in general called modules of elasticity.

In case of longitudinal (tensile or contractile)

$$\frac{\text{Longitudinal stress}}{\text{Longitudinal strain}} = Y \text{ ; the young's modulus.}$$

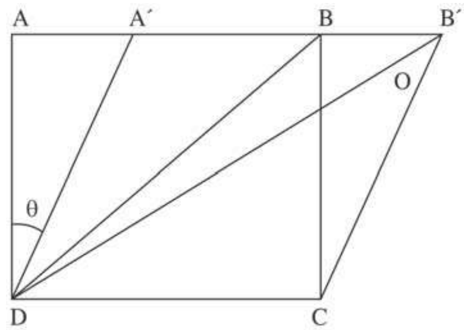
In case of volume

$$\frac{\text{Volume stress}}{\text{Volume strain}} = K \text{ ; Bulk modulus}$$

In case of Shear

$$\frac{\text{Shearing strain}}{\text{Shearing strain}} = \eta \text{ ; rigidity modulus.}$$

A cube ABCD of each side L is given a shear θ as shown in the figure. The diagonal DB changes to DB$^/$ and the elongation is DB$^/$ – DB = OB$^/$



$$= \frac{OB'}{BB'} \times BB'$$

$$= \frac{1}{\sqrt{2}} \frac{BB'}{BC} . BC$$

$\because \quad \lfloor BB'O \approx 45^0$

$$= \frac{L}{\sqrt{2}} \theta \qquad \theta = \text{Shear} = \frac{BB'}{BC} = \frac{AA'}{AD}$$

$\therefore$ Longitudinal tensile strain along DB is

$$\frac{OB'}{DB} = \frac{L\theta}{\sqrt{2}} \times \frac{1}{\sqrt{2}L} \qquad \because \frac{BC}{DB} = \frac{1}{\sqrt{2}}$$

$$= \frac{\theta}{2}$$

Similarly longitudinal contractile strain along AC is also $\frac{\theta}{2}$. AC and DB are perpendicular to each other. Thus a shear is equizalent to two mutually perpendicular strains one tensile and other contractile each of value half the strain and vice versa.

7.3 Poisson Ratio

When ever a body is elongated due to some external force, there is a countraction in the perpendicular direction of the body and similarly counteraction in a direction causes elongation in perpendicular direction. Therefore longitudinal tensile strain is accompanied by lateral contractile strain and vice versa. The ratio of lateral strain to longitudinal strain is known as Poisson ratio. In L and l are the longitudinal and lateral dimension of a body and ΔL and Δl are the changes in the dimensions, then Longitudinal strain = $\frac{\Delta L}{L}$

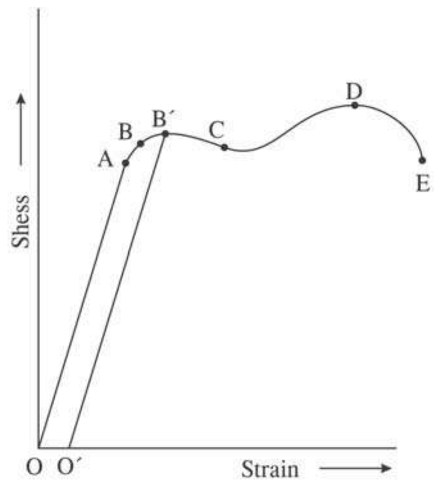
and Lateral strain = $\frac{\Delta l}{l}$

and Poisson ratio is

$$\sigma = \frac{\Delta l}{l} \bigg/ \frac{\Delta L}{L} = \frac{L}{l}\frac{dl}{dL}$$

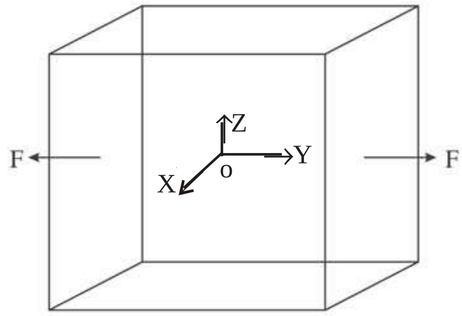
Tensile deformation is considered + ve where as contractile deformation is considered – ve. In order that materials have + ve Poisson ratio the definition of Possion ratio contains a minus sign.

7.4 Load-Elongation / Stress-Stain Relation



One end of a long vertical cylinder of the material under study is fixed at a point in the ceiling. A mass M is hung at the lower end of the cylinder. Slowly but steadily the cylinder elongates, then stops and becomes steady. The external downward force Mg increases the length of the cylinder. If 'L' and 'l' be the original length and increase in length of the cylinder, l/L is the longitudinal strain in the material. The strain produces longitudinal stress in the material of the cylinder. At the steady state the external deforming force per unit area of the cylinder i.e. Mg/A is the measure of the stress, where A is the area of cross section of the cylinder. Loads are gradually increased in small steps and each time the corresponding increase in length is noted. A graph of stress against strain is obtained, which gives an insight of the elastic propertics of the material of the cylinder.

At the beginning the graph is a straight line (OA). The stress is proportional to the strain, showing the validity of Hooke's law. The slope of the graph gives the young's modulus of the material of the cylinder. After that the graph bends down a bit shown by the portion ABC. After A there is always a limiting value of load before which strain totally disappears on removal of the load. If AB is such a portion, B is known as elastic limit. After



B on removal of the load the body does not regain its original size. (Shown by B′O′), instead assumes a new set. After C the body becomes plastic, even when the stress is zero strain is not zero 'C' is called yield point. The point 'D' is the ultimate tensile strength of the material. Hence if any additional strain is produced beyond this point fracture can occur (E). Between the points D and E at a weak point the cylinder becomes narrow known as 'neck' and then the material ruptures. If 'D' and 'E' are close to each other the material is 'brillle' and if they are far apart the materials is 'ductile'.

Relations between elastic constants :

We consider a cube with sides parallel to co-ordinate axes. On the two opposite faces perpendicular to y-axis tensile force F is applied. If L is the length of the side and ΔL is increase of the length due to application of the force.

The longitudinal stress along Y-axis is F/L^2 and the longitudinal strain is $\Delta L/L$.

The Young's modulus is $Y = \dfrac{F/L^2}{\Delta L/L} = \dfrac{F}{L\Delta L}$

$$\Delta L = \frac{F}{YL} \qquad (1)$$

X- and Z- dimension will decrease by Δl, (say). Then lateral strain is $\Delta l/L$ and the poisson ratio is

$$\sigma = \frac{\Delta l/L}{\Delta L/L} = \frac{\Delta l}{\Delta L} \qquad (2)$$

or $$\Delta l = \sigma\, \Delta L = \frac{\sigma F}{Y\, L} \qquad (3)$$

If $(F - F)$ forces are simultaneously applied along X–, Y– and Z– directions, each side will increase by $(\Delta L - 2\Delta l)$.

The volume $V = L^3$ of the cube will increase by $\Delta V = (L + \Delta L - 2\Delta l)^3 - L^3$

or $$\frac{\Delta V}{L^3} = \left(1 + \frac{\Delta L - 2\Delta L}{L}\right)^3 - 1$$

or $$\frac{\Delta V}{V} = 1+3\,\frac{\Delta L-2\Delta L}{L}+\ldots\ +\ldots-1$$

$$=\frac{3}{L}\left(\Delta L-2\Delta L\right)= \text{Volume Strain}$$

According to definition, the bulk modulus is

$$K=\frac{F/L^2}{\frac{3}{L}\left(\Delta L-2\Delta l\right)}=\frac{F}{3L\left(1-\frac{2\Delta l}{\Delta L}\right)\Delta L}$$

$$=\frac{F}{3L(1-2\sigma)}\cdot\frac{YL}{F}\ \text{from (1) and (2)}$$

$$=\frac{Y}{3(1-2\sigma)}$$

or $$Y=3K(1-2\sigma) \tag{4}$$

Application of ($F-F$) force along Y-direction causes Y direction elongation $\Delta L = F/YL$ and X– and Z – direction contraction given by $\Delta L=\frac{\sigma F}{YL}$ as shown above.

Application of (F – F) contractile force along X – direction cause contraction $\Delta L = F/YL$ along X and elongation by

along Y and Z direction given by $\Delta l=\frac{\sigma F}{YL}$.

If the tensile force ($F-F$) along Y and contractile force ($F-F$) along X are applied simultaneously X dimension will decrease by ($\Delta L + \Delta l$) and the strain along this direction is

$$\frac{\Delta L+\Delta l}{L}=\frac{F/YL+\sigma F/YL}{L} = \frac{F}{YL^2}(1+\sigma)$$

This strain is contractile. The Y-direction will increase by same amount and the tensile strain along Y- is

$$\frac{F(1+\sigma)}{YL^2}.$$

We know a pair of equal longitudinal strains one tensile and other contractile along mutually perpendicular direction is equivalent to a shear of value twice the either longitudinal strain. Thus in our case the shear is given by

$$\theta = \frac{2F(1+\sigma)}{YL^2}$$

and rigidity modulus η is given by

$$\eta = \frac{\text{Shearing Stress}}{\text{Sheming Strain}} = \frac{F/L^2}{2\,F(1+\sigma)/YL^2}$$

or $\quad Y = 2\eta(1+\sigma) \qquad (5)$

Equation (4) and (5) can be solved for other two relations as

$$K = \frac{2\eta(1+\sigma)}{3(1-2\sigma)} \qquad (6)$$

and $$Y = \frac{9K\eta}{\eta+3K} \qquad (7)$$

from relations (5) and (6) the Poisson ratio may be expressed in terms of other elastic constants as

$$\sigma = \frac{Y}{2\eta} - 1 \qquad (8)$$

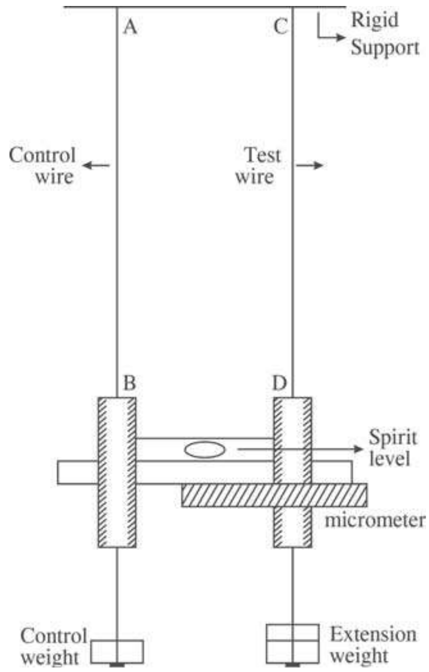
an $$\sigma = \frac{3K-2\eta}{6K+2\eta} \qquad (9)$$

Limiting values of Poisson ratio. Relation (4) suggests that $\sigma < .5$ and relation (5) suggests that $\sigma > -1$.

Thus theoretically $-1 < \sigma < .5$. But the material to have only + ve value of σ we write $0 < \sigma < .5$.

7.5 Determination Young's modules by Searle's method

Two wires, a control wire AB and a test wise CD. Of equal length are attached to a rigid support at the upper ends. The lower ends of the wires are attached to a horizontal bar supporting a spirit level. The bar is hinged to the control wire so that when the test wire



is extended on increasing weight, the spirit level gets tilted by small amount. By turning the micrometer attached to the test wire the spirit level is brought back to its original position. By measuring the amount of turning of the micrometer the increase in length of the test wire due to increase of the weight is obtained. The weight on the test wire side is increased in steps and each time the extension '*l*' is measured by adjusting the micrometer. A graph of load against extension is drown from the reading.

The youngs modulus is given by

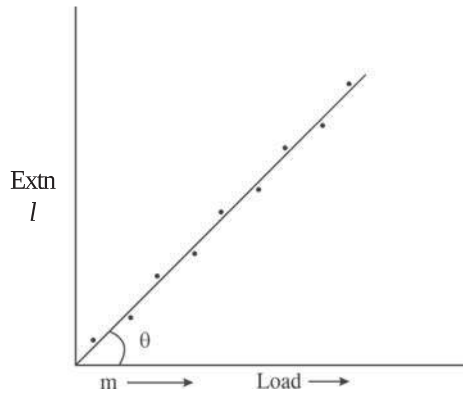
$$Y = \frac{\text{Stress}}{\text{Strain}}$$

or $$Y = \frac{F/A}{l/L}$$

$$= \frac{FL}{\pi r^2 l}$$

$$= \frac{gL}{\pi r^2} \frac{m}{l}.$$

or $$Y = \frac{gL}{\pi r^2} \frac{1}{\tan\theta}$$

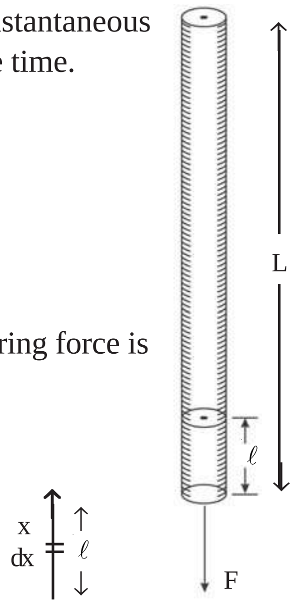


F = mg, the force
A = π r^2 = area of crossection of the wire.
r = radius of the wire
L = length of the experiment wire.

L is measured with the help of a meter scale. *r* is measured with a screw gauge. Measuring the slope of the curve young's modulus is obtained from the above expression.

7.6 Work done in streaching a wire

We consider a vertical wire of length L and area of cross section A fixed at upper end. A downward force F is applied at the lower end. The wire is elongated by *l* (say). The



process of elongation is slow and takes a time. Let x is the instantaneous elongation ($x < l$) and the restoring force is f at an intermediate time.

$$Y = \frac{F/A}{x/L} = \frac{f\ L}{Ax}$$

or $$f = \frac{YAx}{L}$$

In streaching further by dx the work done against the restoring force is

$$fdx = \frac{YA}{L} xdx$$

The work done to perform the complete elongation l is

$$W = \int_0^l f\ dx = \int_0^L \frac{YA}{L} x\ dx$$

$$= \frac{YA}{L}\frac{l^2}{2} = \frac{1}{2} Fl \qquad \because F = \frac{YAl}{L}$$

7.7 Strain energy

The work done by the external force in streching the wire is stored in the wire as potential energy and is known as strain energy of the wire. Thus the strain energy say 'U' is given by

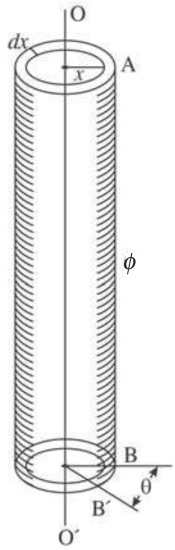
$$U = \frac{1}{2} F.l. = \frac{1}{2} \times \frac{F}{A} \times \frac{l}{L} \times AL$$

$$= \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume.}$$

Thus the strain energy density of a stretched wire is equal to $\frac{U}{V} = \frac{1}{2}$ (stress × strain)

7.8 Twist of a wire or a cylinder

We consider a wire / cylinder of length 'L' and radius 'r'; whose upper end is fixed and a twist θ is applied at the lower end. In the figure a co-axial cylindrical cell of redii between x and x + dx (0 < x < r) is considered within the wire. AB is a vertical line on the surface of the cylindrical cell before application of the twist. On application of the twist θ at the bottom B moves to B' (say). If ϕ is the shear $BB^{/} = L\phi = x\ \theta$



or $$\phi = \frac{\theta}{L}x$$

The shearing stress on the annular region at the bottom of the wire is given by stress = shearing strain × rigidity modulus.

$$= \eta\phi = \eta\frac{\theta}{L}x$$

The restoring force on the annular region opposing the twist is equal to the shearing stress multiplied by the area.

$\therefore$ The restoring force $= \dfrac{n\theta}{L}x \times 2\pi\ xdx$

$$= \frac{2\pi\eta\theta}{L}x^2dx$$

The moment of this force about the axis $00^{/}$ of the wire is

$$\frac{2\pi\eta\theta}{L}x^3dx$$

The moment of total restoring force also known as twisting coupe is

$$\Gamma = \frac{2\pi\eta\theta}{L}\int_0^r x^3\ dx \qquad = \frac{2\pi\eta\theta}{L} \times \left|\frac{x^4}{4}\right|_0^r$$

or $$\Gamma = \frac{\pi\ \eta\ r^4}{2\ L}\theta$$

or $$\Gamma = C\theta$$

where $C = \dfrac{\pi\ \eta\ r^4}{2\ L}$, is known as moment of restoring couple per unit twist. This also known as tortional rigidity of the wire.

Work done in twisting a wire.

We consider a wire of length 'L' and radius '*r*' fixed at one end and a torque Γ is applied to the other end to produce a twist θ. An equal and opposite restoring torque will be produced in the wire at equilibrium.

$$\Gamma = \frac{\pi \eta r^4}{2L}\theta$$

or $\quad \Gamma = C\theta, \quad C = \frac{\pi \eta r^4}{2L} =$ a constant for the wire known as restoring torque per unit twist.

The twist takes a time to allend the final value θ. Let α is an instantaneous value of twist at an intermediate time and τ is the corresponding restoring torque

$$\tau = c\,\alpha$$

For a further twist $d\alpha$, the work done is

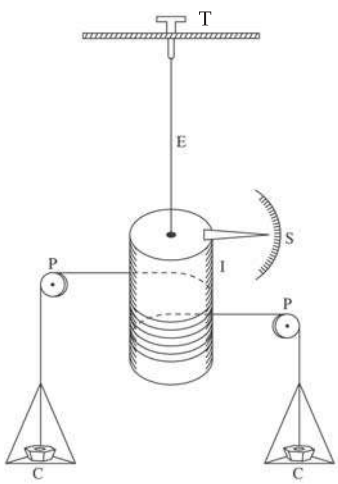
$$dW = \tau\, d\alpha$$

The total work done by the external torque against the restoring torque to produce twist θ is

$$W = \int_0^\theta \tau\, d\alpha = C\int_0^\theta \alpha\, d\alpha = \frac{1}{2}c\,\theta^2 = \frac{1}{2}\Gamma\theta$$

7.9 Rigidity modulus by static torsion

A cylinder of diameter *d* is hung from a torsion head T with the help of a wire 'E' of the material whose rigidity modulus is to be determined. Let 'L' and 'r' be the length & radius of the wire. Two ends of a long thread after spiraling on the cylinder hold two pans C – C via two small pulleys P – P. If weights are placed on the pans the thread unwinds and gives rotation to the cylinder and twist to the specimen wire. An indicator I is fixed on the top of the cylinder which can move over a circular scale 'S' and reads the rotation of the cylinder and the twist 'θ' of the wire. Equal weights are placed on the pans and the twist is noted. The weights are increased in steps and each time the reading of the indicator



is taken. A graph of the twist 'θ' against the mass 'm' placed on the pans is drawn. Average value of m/θ is obtained from the graph and the value of rigidity modulus is calculated from the relation below. The external torque twisting the wire is

$$mg \times d$$

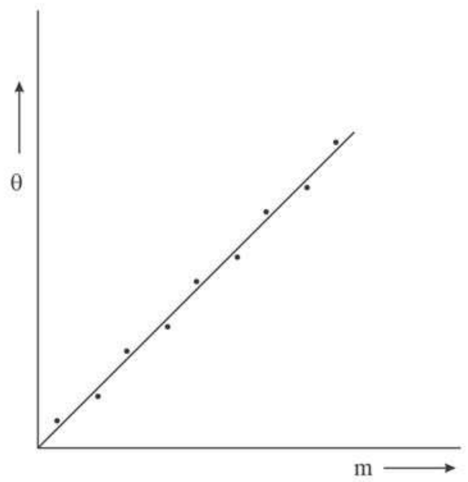
and the internal restoring torque produced is

$$\frac{\pi \eta r^4}{2L} \theta .$$

At equilibrium

$$mgd = \frac{\pi \eta r^4}{2L} \theta$$

or $$\eta = \frac{2g\, dL}{\pi r^4} \frac{m}{\theta}$$



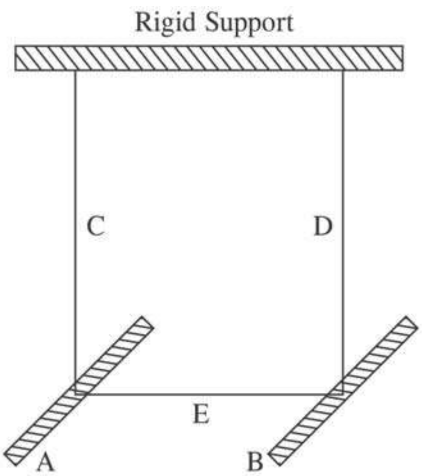
'L' is measured by a meter scale

'd' is measured by a slide calipers.

'r' is measured by a screw gauge.

7.10 Elastic constants by Searles method

The following experiment enables us to obtain all the elastic constants. Two identical bars A and B of square or circular cross-section are connected together at their middle points by a wire E of the material whose elastic constants are to be measured. The bars are suspended by two silk fibres C & D from a rigid support such that the bars A and B and the wire E are in same horizontal plane.



Mass 'M' of each bar is measured with the help of a physical balance. The length 'L' of the bars is measured by a meter scale. For bars of square cross section breadth – 'b' or for the bars of circular cross-section the radius 'r' is measured by a slide calipers.

The length 'l' and radius 'r' of the speciman E are measured by a metre scale and screw gange respectively.

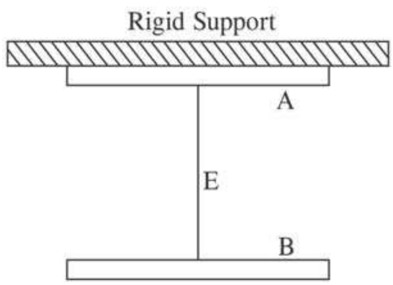
The nearer ends of the bars A and B are brought closer with the help of a small thread. The thread is burned. The bars will move to and fro about the silk suspension. The time period of oscillation is measured. It T_1 is the time period and I is the moment of inertia of the bars about the suspension, the Young's modulus of the material of wire E is

$$Y = \frac{8\pi\, Il}{T_1^2\, r^4}$$

$$I = \frac{M}{2}\left(L^2 + b^2\right) \text{ : for square cross section}$$

$$= M\left(\frac{L^2}{12} + \frac{R^2}{4}\right) \text{ : for circular cross section}$$

Next the silk fibres C and D are removed. One of the bars say A is attached to the rigid support, such that the specimen wires is vertical and bar B can be made to oscillate about the specimen wire. If T_2 be the time period of oscillation of B, the rigidity modulus of the material of specimen is



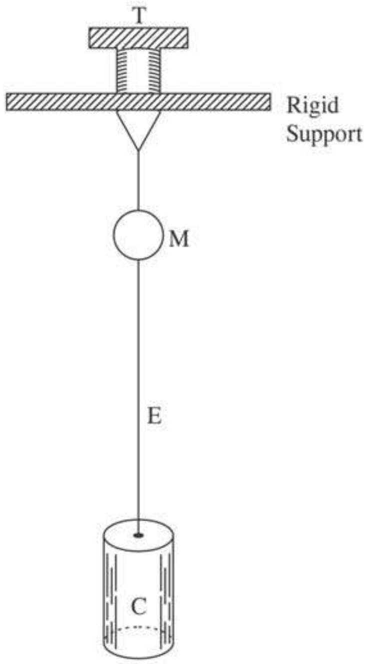
$$\eta = \frac{8\,\pi\, Il}{T_2^2\, r4}$$

The poisson ratio is $\sigma = \dfrac{T_2^2}{2T_1^2} - 1$ and the bulk modulus is

$$K = \frac{Y}{3(1-2\sigma)}$$

7.11 Rigidity modulus by dynamic method

A wire 'E' of the material, whose rigidity modus is to be determined is hung from the torsion head T in a rigid support. At the lower end of the wire a cylinder C is attached.



If the cylinder is given a small rotation by an external torque, the wire gets twisted. The moment of the restoring couple is equal and opposite to moment of the twisting couple. The cylinder's simple harmonic motion for small θ is governed by the equation of motion

$$I\ddot{\theta} = -c\theta$$

where c is the moment of restoring couple per unit twist of the wire.

or $$\ddot{\theta} + \omega^2\theta = 0, \quad \omega^2 = \frac{c}{I}$$

or $$\left(\frac{2\pi}{T}\right)^2 = \frac{c}{I} \quad \text{or} \quad T^2 = \frac{4\pi^2 I}{c},$$

T = time period of Oscillation

We know $c = \dfrac{\pi \eta r^4}{2L}$

'r', the radius of the wire and is measured with the help of a screwgauge, 'L', the length of the wire, measured with the help of a meter scale.

From the above relation we get

$$c = \frac{4\pi^2}{T^2} \times I$$

$$= \frac{4\pi^2}{T^2} \times \frac{1}{2} MR^2$$

M is the mass of the cylinder and R is the radius of the cylinder.

or $$\frac{\pi \eta r^4}{2L} = \frac{2\pi^2 MR^2}{T^2}$$

or $$\eta = \frac{4\pi MR^2 L}{T^2 r^4}$$

with the help of a stopwatch and the minor M the time period of oscillation can be measured.

7.12 Moment of inertia by torsion balance

The experimental arrangement consists of a circular table known as inertia table D fitted with a pair of small vertical pillars P–P and a horizonal bar B. The whole assembly is suspended by a thin wire W from the torsion head T inside a rigid frame F. The frame is mounted on a heavy iron base. A mirror M is used for the measurement of time period of oscillation. The entire apparatus is enclosed in a glass cover. The table D is set into torsional vibration and the time period (t_0) is precisely measured, which is related to the moment of inertia I_0 of inertia table an restoring couple per unit twist c of the wire W by

$$t_0 = 2\pi\sqrt{\frac{I_0}{C}} \qquad (1)$$

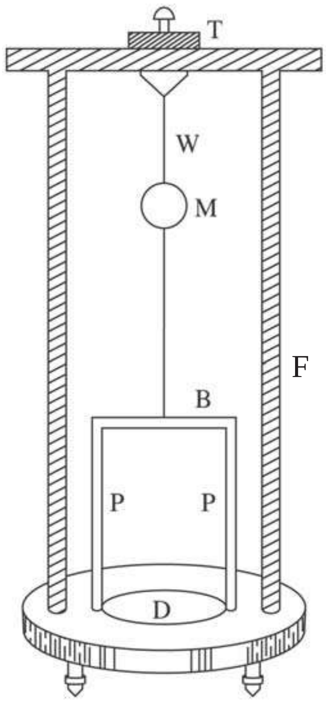
An object whose moment of inertia I is to be measured is placed on the inertia table. Again the inertia table is set into torsional vibration. If 't' by the time period of oscilation,

$$t = 2\pi\sqrt{\frac{I+I_0}{C}} \qquad (2)$$

The experimental body is replaced by another body of known moment of inertia say I´, the time period of oscillation will be given by

$$t' = 2\pi\sqrt{\frac{I'+I_0}{C}}$$

(3)



Solving for I, from equation (1), (2) and (3) we get

$$I = I'\frac{t^2 - t_0^2}{t'^2 - t_0}$$

Perfectly rigid body : A body will be called perfectly rigid body if it does not suffer any deformation whatever be the external deforming force. No body is perfectly rigid.

Perfectly elastic body : Whatever be the magnitude of deforming force, if a body completely regains its shape and size the body is said to be perfectly elastic. No body is perfectly elastic.

Perfectly plastic : A body is said to be perfectly plastic if it retains its changed configuration even after removal of external force.

Problem 1 : – A piece of copper wire has twice the radius of a steel wire. One end of the copper wire is joined to the one end of the steel wire so that both can be subjected to the same longitudinal deforming force. By what fraction of its length will the steel wire be stretched when the length of the copper wire has increased by 0.5 %. Y for steel is twice that of the copper. We know that

$$Y = \frac{F}{\pi r^2} \Big/ \frac{\Delta L}{L}$$

For steel

$$Y_s = \frac{F}{\pi\, r_s^2} \times \frac{L_s}{\Delta L_s} \tag{1}$$

For Copper

$$Y_c = \frac{F}{\pi\, r_c^2} \times \frac{L_c}{\Delta L_c} \tag{2}$$

Dividing eqn (1) by eqn (2) we have

$$\frac{Y_s}{Y_c} = \frac{r_c^2}{r_s^2} \times \frac{L_s}{\Delta L_s} \times \frac{\Delta L_c}{L_c}$$

or $$2 = 4 \times \frac{L_s}{\Delta L_s} \times \frac{0.5}{100}$$

or $$\frac{\Delta L_s}{L_s} = \frac{4}{2} \times \frac{0.5}{100} = \frac{1}{100}$$

Problem 2 :

A load of 24 kg is suspended from one end of a wire whose radius is 1 mm. What will be the change of temperature of the wire if the wire suddenly snaps ? For the material of the wire Y = 12 × 10^{11} dyn.cm^{-2}, density = 9 gm cc^{-1} sp.heat = 0.1 and J = 4.2 × 10^7 ergs cal^{-1}.

Solution : We know

$$Y = \frac{mg}{\alpha}\bigg/\frac{\Delta L}{L}$$

or $$\Delta L = \frac{mgL}{\alpha Y}$$

$$= \frac{24\times 10^3\times 980\times L}{\pi\times 0.01\times 12\times 10^{11}}$$

$$= \frac{2L\times 980}{\pi\times 106} = \frac{196L}{\pi\times 10^5}$$

Work done

$$W = \frac{1}{2}\,\Delta L.F = \frac{1}{2}\times\frac{196L}{\pi\times 105}\times 24\times 10^3\times 980$$

$$= \frac{196\times 12\times 98\times L}{\pi\times 10}$$

Heat

$H = ms\theta$ $\qquad\qquad$ θ = rise of temperature

$= \pi\times(0.1)^2 L\times 9\times 0.1\times\theta$

$= 9\pi L\theta\times 10^{-3}$

Again

$W = JH$

or, $$\frac{196\times 12\times 98\times L}{\pi\times 10} = 4.2\times 10^7\times 9\pi L\theta\times 10^{-3}$$

or $$\theta = \frac{196 \times 12 \times 98}{\pi^2 \times 10 \times 4.2 \times 10^7 \times 9 \times 10^{-3}}$$

$= 0.0061\ ^{o}C$

Problem 3 : A metal wire of length 3 m and diameter 2 mm is stretched by 10 kg-wt. Find the contraction of the diameter. $Y = 20 \times 10^{11}$ dyn.cm^{-2} and $\sigma = 0.26$

Solution : $Y = \frac{F}{\alpha} \Big/ \frac{\Delta L}{L}$, or $\frac{\Delta L}{L} = \frac{F}{\alpha Y}$

$\sigma = \frac{\Delta D}{D} \Big/ \frac{\Delta L}{L}$ or $\frac{\Delta L}{L} = \frac{\Delta D}{\sigma D} = \frac{F}{\alpha Y}$

$$\therefore\ \Delta D = \frac{\sigma DF}{\alpha Y} = \frac{0.26 \times 0.2 \times 10000 \times 980}{\pi \times 0.01 \times 20 \times 10^{11}}$$

$= 8 \times 10^{-6}$ cm.

9.13 Exercises

1) A wire 2m long and 2.0 mm in diameter stretched by 8 kg. The length increases by 0.24 mm. Find stress, strain and Y.

2) A wire is stretched by the application of a force of 50 kg-wt per square cm. What is the percentage increase in length $Y = 7 \times 10^{10}$ Nm^{-2}

3) A tangential force of 0.245 N is applied on the upper surface of a block (60 mm × 60 mm × 60 mm). The upper surface is displaced by 5 mm with respect to the lower surface. Find shear stress, shear strain and modulus of rigidity.

4) A cylindrical mass 5 kg of radius 4 cm is hung with a metal wire of length 100 cm and radius 2 mm. When the cylindrical mass is made to oscillate it makes 25 oscillations in 20s. Compute the rigidily modulus of the material of the wire.

Unit 8 ❐ Special theory of relativity

Structure

8.1 Objectives

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8.2 Introductions

Einstein in 1905 at the age of 26 yrs. through a paper "On the electrodynamics of moving bodies" introduced special theory of relativity. The inconsistancy of Newtonian mechanies with Maxwell's equations of electromagnetism and the failure of experimental proof of existance of a hypothesided luminiferous ether led Einstein to propose the theory based on two postulates.

1. The physical laws are invariant in all inertial system.

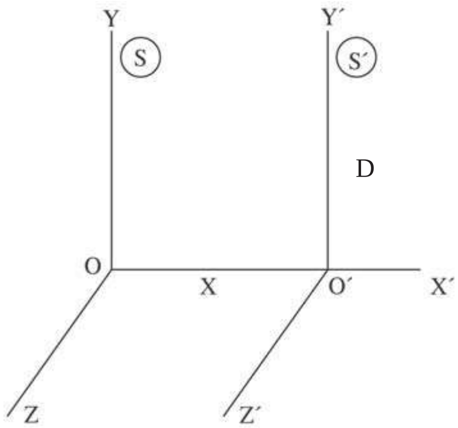
2. The speed of light in vacuum is the same for all observers regardless of motion of the light source.

The propagation of waves could not be entertained without a medium at the time light was initially thought to be wave in nature. An omnipresent hypothetical medium called eather was required the light to creep through. As time passed the enrichment of scientific knowledge took place and ether became more and more self contradictory. Though none was in favour of ether hypothesis everybody had to swallo it for absence of suitable alternative physical theory.

Michelson – Morley's Experiment (1887) failed to find any measurable property of ether and the null result of the above experiment was only explained with the help of Einstein's postulates, which laid the foundation of special theory of relativily.

8.3 Lorenz transformation

Einsteins second postulate of constancey of velocity of light was a blow to the Galilean transformation laws in Newtonian mechanics.



Let S and S$'$ are two inertial frames, S is at rest and S$'$ is moving along common X-axis with uniform velocity *v*. If P is a point whose space and time coordinates are (x, y, z, t) and (x$'$, y$'$, z$'$, t$'$) inS and S$'$ frames respectively, then Galilean transformation equations are x = x$'$ + vt ; y = y$'$, z = z$'$ and t = t$'$.

We suppose an observer in s and light is moving in the frame s$'$ with velocity c along x-axis (x & x$'$ axis coincident) Differentiating galilean transformation equation with respect to time we get

$$\frac{dx}{dt} = \frac{dx'}{dt} + v$$

$$= c + v > c$$

The velocity of light appears to be $c + v$ to the observer in *s* frame, which is not possible according to the second postulate of Einstein. Thus Galilean transformations need modification. Lorentz derived the new set of transformation equations as

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or Inverse Lorentz transformatin as

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

While deriving the above relations Lorenlz kept the following facts in his thinking.

1) Constancy of velocity of light

2) New relations must boil down to Galilean relations in the limit v << c.

3) The relations must be linear in space and time.

8.4 Length Contraction

We imagine a rod lying at rest along $x^{/}$ axis in $S^{/}$ frame. Its end points are measured to be at x_2' and x_1'. So that the rest length is $x_2' - x_1' = L_0$ say.

In order to measure the length of the rod from S-frame we must read the end positions X_2 and X_1 simultaneously say at a time t.

Using Lorentz transformation

$$x_2' = \frac{x_2 - vt}{\sqrt{1-v^2/c^2}}, \quad x_1' = \frac{x_1 - vt}{\sqrt{1-v^2/c^2}}$$

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-v^2/c^2}}$$

Thus the length measured from s frame is

$$x_2 - x_1 = (x_2' - x')\sqrt{1-v^2/c^2}$$

or $$L = L_0\sqrt{1-v^2/c^2}$$

Thus a body's length is measured to be greatest when the body is at rest relative to the observer. When it moves with a velocity v with respect to the observer its measured length is contracted in the direction of its motion by the factor $\sqrt{1-v^2/c^2}$, where as its dimensions perpendicular to the direction of motion are unaffected.

The frame in which the observed body is at rest is known as proper frame and the length of the rod in such a frame is known as proper length.

8.5 Time dialation

We consider a clock to be at rest at the position x' in the S'-frame. Let t'_1 and t'_2 are times recorded by the clock at x' in s'. The times will be recorded as t_1 and t_2 say from s-frame. Using Lorentz inverse transformation we get.

$$t_1 = \frac{t_1' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}$$

and $$t_2 = \frac{t_2' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}$$

$$\therefore\ t_2 - t_1 = \frac{t_1' - t_2'}{\sqrt{1 - v^2/c^2}} \qquad\qquad t_2 - t_1 > t_2' - t_1'$$

Therefore from the point of view of observer S, the moving S$^{/}$-clock appears slowed down by a factor $\sqrt{1-v^2/c^2}$. The proper time interval $t_2' - t_1' = d\tau$ (say) is the time interval recorded by a clock attached to the observed body. A non proper time interval $t_2 - t_1^{/}$ = dt (say) would be a time interval measured by two different clocks at two different places, thus

$$dt = \frac{d\tau}{\sqrt{1-v^2/c^2}}$$

8.6 Relativistic velocity addition

We write down the Lorentz transformation equations as

$$x = \frac{x' + vt'}{\sqrt{1-v^2/c^2}}$$

$$y = y'$$

$$z = z'$$

and
$$t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}$$

Taking the differentials we get.

$$dx = \frac{dx' + v\,dt'}{\sqrt{1-v^2/c^2}} \tag{1}$$

$$dy = dy' \tag{2}$$

$$dz = dz' \tag{3}$$

and
$$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-v^2/c^2}} \tag{4}$$

Dividing equation (1) by equation (4) we get

$$\frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + v/c^2\,dx'} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}$$

or $$u_x = \frac{u'_x + v}{1 + \frac{v\,u'_x}{c^2}} \qquad (5)$$

where u_x and u'_x are velocities of a body as observed from s and s' frames respectively, along common x-axis.

Dividing equation (2) by equation (4)

$$\frac{dy}{dt} = \frac{dy'\sqrt{1-v^2/c^2}}{dt' + v/c^2\,dx'} = \frac{\frac{dy'}{dt}\sqrt{1-v^2/c^2}}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}$$

or $$u_y = \frac{u'y\sqrt{1-v^2/c^2}}{1 + \frac{v\,u'_x}{c^2}} \qquad (6)$$

and similarly

$$u_z = \frac{u'z\sqrt{1-v^2/c^2}}{1 + \frac{v\,u'_x}{c^2}} \qquad (7)$$

u_y and u_z are the y and z component of velocities as observed from s-frame and u'_y and u'_z are the respective velocities as observed from s' frame of the said body.

Problem :

The velocity of light with respect s$'$ framme moving with velocity 0.9 c is. c. If s he the rest frame, what will the velocity of light with respect to s frame. Assume all velocities are along x-asis.

We know

$$u_x = \frac{u'_x + v}{1 + \frac{vu}{c^2} x'}$$

$$= \frac{c + 0.9c}{1 + \frac{0.9c \times c}{c^2}}$$

= c.

Problem :

At an attitude of 10000 m μ-mesons are created with half-life 1.56 μs. None of them are therefore expected to be present on earth surface. But a large number of μ-mesons are present on the earth surface. If the velocity of μ-mesons are taken to be 0.98C, the time taken by the mesons to reach earth

$$T = \frac{10000}{0.98 \times 3 \times 10} = 34\ \mu s$$

$$= \frac{34}{1.56} = 21.8 \text{ halflives}$$

∴ Survival rate = $2^{-21.8} = 0.27 \times 10^{-6}$ mesons are not expected to reach earth. If relativic effect is taken from earth's fame,

$$T = \frac{10000}{0.98 \times 3 \times 10^8} = 34\ \mu s$$

The half life is $1.56\sqrt{1-(0.98)^2}\ \mu s$

= 7.8 μs

$$\therefore\ T = \frac{34}{7.8} = 4.36 \text{ halflives}$$

The survival rate = $2^{-4.36}$ = 0.049

i.e. 49000 out of one million is expected on the earth surface.

If relativistic effects is considered from meson's frame.

$$T = \frac{10000\sqrt{1-(0.98)^2}}{0.98\times 3\times 10^8} = \frac{2000}{98\times 3\times 10^6}$$

$$= 6.8\ \mu\text{s} = \frac{6.8}{1.56} = 4.36 \text{ half life}$$

Survival rate = $2^{-4.36}$ = 0.049 same as when observed from earth's frame. This explains the presence of muon on earth.

8.7 Exercises

(1) A 2 m tall and 50 cm wide cosmonant is approaching earth in a spaceship moving with velocity 0.97c. The length of the cosmonant is parallel to the direction of motion. What are the height and width of the cosmonant according to an observer on earth ?

(2) The average life time of a π meson in its own frame is 26 ns. The meson moves with speed 0.95c with respect to earth. What is the life time with respect to an observer at rest on earth. What is the average distance it travels before decaying as measured by the observer of earth ?

(3) Two spaceships approach each other, each moving with same speed as measured by a stationary observer on earth. Their relative speed is 0.7c. Determine the velocity of each space from earth.

8.8 Suggested Reading

1. Mathematics of Physics and Chemistry.
   By : H. Margenu and G. M. Murphy
2. Mathematical methods for Physicists
   By : George B. Arfken.
3. Vector Analysis
   By : Murray R. Speigel

4. General properties of Matter
   By : D. S. Mathur
5. General properties of Matter
   By : F. H. Newman and V. H. L. Searle
6. Classical Mechanics
   By : Herbert Goldstein
7. Classical Mechanics
   By : R. G. Takwale and P. S. Puranik
8. Physics
   By : D. Halli day and R. Resnick